AM 221: Advanced Optimization	Spring 2016
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1. Taylor expansion Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) = 0$ and $f'(0) > 0$ show that there exists $\varepsilon > 0$ such that f is positive on the interval $(0, \varepsilon]$ and negative on the interval $[-\varepsilon, 0)$.

Proof. By definition of the derivative, $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$. Let us choose $\delta > 0$ such that $f'(0) - \delta > 0$ (such a δ exists since $f'(0) > 0$). By definition of the limit, there exists $\varepsilon > 0$ such that:

$$|h| \leq \varepsilon \Rightarrow \frac{f'(h)}{h} \geq f'(0) - \delta$$

Hence, if $h \in (0, \varepsilon]$, the above inequality implies: $f'(h) \geq h(f'(0) - \delta) > 0$ (since $h > 0$ and $f'(0) - \delta > 0$). Similarly, if $h \in [\varepsilon, 0)$ we have $f'(h) \leq h(f'(0) - \delta) < 0$. □

2. Sequences, Bolzano-Weierstrass Theorem

a. Show that a non-decreasing and bounded above sequence of real numbers is convergent. Give a counter-example when removing the "increasing" assumption.

In the remaining parts of this problem, we will prove the Bolzano-Weierstrass theorem: *every bounded sequence of real numbers has a convergent subsequence*. Recall that a subsequence of a sequence $(u_n)_{n\geq 0}$ is $(u_{\phi(n)})_{n\geq 0}$ for some increasing function $\phi : \mathbb{N} \to \mathbb{N}$. Let us consider a bounded sequence $(u_n)_{n\geq 0}$ and denote by a and b two real numbers such that $a \leq u_n \leq b$ for all n.

b. Show that at least one of the intervals $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$ contains infinitely many terms of $(u_n)_{n\geq 0}$.

c. Repeat the above argument to construct two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of real numbers and an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that:

   – (a_n) is non-decreasing, (b_n) is non-increasing.
   – $\lim_{n\to\infty} b_n - a_n = 0$.
   – $a_n \leq u_{\phi(n)} \leq b_n$ for all n.
   – $S_n = \{m \in \mathbb{N} : m > \phi(n) \text{ and } a_n \leq u_m \leq b_n\}$ is infinite.

d. Conclude.

Proof of c. and d. We will construct a_n, b_n and $\phi(n)$ inductively.

Base case: for $n = 0$ we define $a_0 = a$, $b_0 = b$ and $\phi(0) = 0$.

Inductive step: assume we have constructed a_n, b_n and $\phi(n)$ satisfying all the properties of part c. Consider the two intervals $[a_n, \frac{a_n+b_n}{2}]$ and $[\frac{a_n+b_n}{2}, b_n]$. Similarly to part a., since S_n is infinite, one of these two intervals contains infinitely many u_m with $m \in S_n$. Let us denote this interval by I and define $\phi(n+1)$ to be the smallest index in $S_n \cap I$. If $I = [a_n, \frac{a_n+b_n}{2}]$ let us also define $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$, otherwise let us define $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$. In both cases, we have:

- $\phi(n+1) > \phi(n)$ since $\phi(n+1)$ is in S_n.
- S_{n+1} is infinite by choice of I, a_{n+1} and b_{n+1}
- $a_{n+1} \leq u_{\phi(n+1)} \leq b_{n+1}$ since $u_{\phi(n+1)}$ is in I.
- $a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n$ by construction.
- $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$

The last property implies that $b_n - a_n = \frac{b_0 - a_0}{2^n}$ (it is easy to prove by induction). Hence $\lim_{n\to\infty} b_n - a_n = 0$.

We can then conclude that $u_{\phi(n)}$ converges by application of the Squeeze theorem. □

3. Closed sets, bounded sets A *closed set* is a set which contains its limit points. In other words, C is a closed set of $\mathbb{R}^n$ iff for any sequence $(\mathbf{u}_n)_{n\geq 0} \in C^{\mathbb{N}}$ converging to $\boldsymbol{\ell} \in \mathbb{R}^n$ we have $\boldsymbol{\ell} \in C$.

A *bounded set* B of $\mathbb{R}^n$ is a set contained in a ball $B_2(\mathbf{x}, r)$ for some $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. Recall that the ball $B_2(\mathbf{x}, r)$ is defined by:

$$B_2(\mathbf{x}, r) \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\|_2 \leq r\}$$

a. Show that the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is closed. Give a counter-example when f is not continuous. Recall the graph $G_f \subset \mathbb{R}^2$ of f is defined by:

$$G_f = \big\{\big(x, f(x)\big) \mid x \in \mathbb{R}\big\}$$

b. Show that the graph of continuous function $f : [a, b] \to \mathbb{R}$ is a bounded set.

4. Maximum Coverage Problem The maximum coverage problem is a classical problem in combinatorial optimization. In this problem there is a universe of elements $\mathcal{U} = \{1, \ldots, m\}$ and you are given as input a collection of n subsets $S_1, \ldots, S_n$ of $\mathcal{U}$ ($S_i \subseteq \mathcal{U}$) and a budget $k \in \mathbb{N}$. The goal is select a collection $\mathcal{S}$ of at most k of the sets $S_1, \ldots, S_n$ such as to maximize the number of elements of $\mathcal{U}$ contained in the union of the sets in $\mathcal{S}$. In other words, the goal is to solve:

$$\max_{|\mathcal{S}| \leq k} \left| \bigcup_{S_i \in \mathcal{S}} S_i \right|$$

In contrast with optimization problems as introduced in class, the optimization variable $\mathcal{S}$ belongs to the power set of $\{S_1, \dots, S_n\}$ which makes the optimization problem hard to analyze. The goal is this problem is to reformulate the maximum coverage problem in several ways and discuss the relative advantages of the different formulations.

a. Let us introduce one variable $y_j \in \{0, 1\}$, $1 \le j \le m$ for each element of the universe, and one variable $x_i \in \{0, 1\}$, $1 \le i \le n$ for each subset S_i. Write a constraint (inequality) which expresses that $y_j = 0$ if:

$$\forall i \text{ s.t. } j \in S_i, \quad x_i = 0$$

Formulate the maximum coverage problem as an optimization problem on $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$. An optimization problem in this form is known as a 0/1 integer program.

b. Show that the previous optimization program is equivalent to:

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{j=1}^{m} \min\left\{1, \sum_{i: j \in S_i} x_i\right\} \\ \text{s.t} \quad & x_i \in \{0, 1\}, \quad 1 \le i \le n \\ & \sum_{i=1}^{n} x_i \le k \end{aligned}$$

What are the advantages or disadvantages of this formulation compared to the previous one?

c. Show that the previous optimization program is equivalent to:

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{j=1}^{m} \left(1 - \prod_{i: j \in S_i} (1 - x_i)\right) \\ \text{s.t} \quad & x_i \in \{0, 1\}, \quad 1 \le i \le n \\ & \sum_{i=1}^{n} x_i \le k \end{aligned}$$

d. An important technique in combinatorial optimization is to consider *relaxations* of combinatorial problems into continuous problems. For a 0/1 integer program, a natural relaxation is to replace constraints of the form $x_i \in \{0, 1\}$ into constraints of the form $0 \le x_i \le 1$ (x_i is now a real number). Are the continuous relaxations of all the above formulations equivalent? Which one(s) is/are convex optimization programs?