AM 221: Advanced Optimization	Spring 2016
Prof. Yaron Singer	*Section 3 — Wednesday, Feb. 10th*

1. Taking duals. $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$. $\mathbf{a}_i \in \mathbb{R}^m$ is the ith row of A. $A_j \in \mathbb{R}^n$ is the jth column of A.

$$\begin{array}{rlcrl} \min_{\mathbf{x} \in \mathbb{R}^m} & \mathbf{c}^\intercal \mathbf{x} & & \max_{\mathbf{y} \in \mathbb{R}^n} & \mathbf{b}^\intercal \mathbf{y} \\ \text{s.t} & \mathbf{a}_i^\intercal \mathbf{x} \geq \mathbf{b}_i,\ i \in M_1 & & \text{s.t} & \mathbf{y}_i \geq 0,\ i \in M_1 \\ & \mathbf{a}_i^\intercal \mathbf{x} \leq \mathbf{b}_i,\ i \in M_2 & & & \mathbf{y}_i \leq 0,\ i \in M_2 \\ & \mathbf{a}_i^\intercal \mathbf{x} = \mathbf{b}_i,\ i \in M_3 & \Leftrightarrow & & \mathbf{y}_i \text{ free},\ i \in M_3 \\ & \mathbf{x}_j \geq 0,\ j \in N_1 & & & \mathbf{y}^\intercal A_j \leq \mathbf{c}_j\ j \in N_1 \\ & \mathbf{x}_j \leq 0,\ j \in N_2 & & & \mathbf{y}^\intercal A_j \geq \mathbf{c}_j\ j \in N_2 \\ & \mathbf{x}_j \text{ free},\ j \in N_3 & & & \mathbf{y}^\intercal A_j = \mathbf{c}_j\ j \in N_3 \end{array}$$

2. Separation of Polyhedra. Let us consider two polyhedra, $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ and $Q = \{\mathbf{x} \in \mathbb{R}^n \mid C\mathbf{x} \leq \mathbf{d}\}$, for $A, B \in \mathbb{R}^{m \times n}$ and $\mathbf{b}, \mathbf{d} \in \mathbb{R}^m$.

a. Write a linear program which finds $\mathbf{x} \in P \cap Q$ if $P \cap Q \neq \emptyset$ and which is infeasible and $P \cap Q = \emptyset$.

b. Write the dual of the program found in part a.

c. Show that the polyhedra P and Q have an empty intersection if and only if there exists a hyperplane which separates them strictly, *i.e.* there exists $\mathbf{c} \in \mathbb{R}^n$ such that:

$$\mathbf{c}^\intercal \mathbf{x} < \mathbf{c}^\intercal \mathbf{y}, \quad \mathbf{x} \in P,\ \mathbf{y} \in Q$$

3. Vertex Cover, Graph Matching. In the *minimum vertex cover* problem we are given a graph $G = (V, E)$ and the goal is to select a subset $S \subseteq V$ of vertices of minimal size, such that every edge is adjacent to at least one vertex in S, *i.e.* $u \in S$ or $v \in S$ for each edge $\{u, v\} \in E$.

a. Write a 0/1-integer program for the minimum vertex cover problem. Write an LP relaxation for this program.

b. Derive the dual of the LP found in a.

c. Find a combinatorial problem of which the LP found in b. is an LP relaxation.

4. Extreme Points. Given a polyhedron $P \subset \mathbb{R}^n$. Show that $\mathbf{x}$ is an extreme point of P if and only if $\mathbf{x}$ is a basic feasible solution of P.