

Submodular Function Maximization via the Multilinear Relaxation and Contention Resolution Schemes*

Chandra Chekuri[†] Jan Vondrák[‡] Rico Zenklusen[§]

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Abstract

We consider the problem of maximizing a non-negative submodular set function $f : 2^N \rightarrow \mathbb{R}_+$ over a ground set N subject to a variety of packing type constraints including (multiple) matroid constraints, knapsack constraints, and their intersections. In this paper we develop a general framework that allows us to derive a number of new results, in particular when f may be a *non-monotone* function. Our algorithms are based on (approximately) maximizing the multilinear extension F of f [6] over a polytope P that represents the constraints, and then effectively rounding the fractional solution. Although this approach has been used quite successfully [7, 33, 36, 15, 4], it has been limited in some important ways. We overcome these limitations as follows.

First, we give constant factor approximation algorithms to maximize F over a down-closed polytope P described by an efficient separation oracle. Previously this was known only for monotone functions [49]. For non-monotone functions, a constant factor was known only when the polytope was either the intersection of a fixed number of knapsack constraints [36] or a matroid polytope [50, 43]. Second, we show that *contention resolution schemes* are an effective way to round a fractional solution, even when f is non-monotone. In particular, contention resolution schemes for different polytopes can be combined to handle the intersection of different constraints. Via LP duality we show that a contention resolution scheme for a constraint is related to the *correlation gap* [2] of weighted rank functions of the constraint. This leads to an optimal contention resolution scheme for the matroid polytope.

Our results provide a broadly applicable framework for maximizing linear and submodular functions subject to independence constraints. We give several illustrative examples. Contention resolution schemes may find other applications.

1 Introduction

We consider the meta-problem of *maximizing* a non-negative submodular set function subject to independence constraints. Formally, let N be a finite ground set of cardinality n , and let $f : 2^N \rightarrow \mathbb{R}_+$ be a submodular set function over N .¹ Let $\mathcal{I} \subseteq 2^N$ be a downward-closed family² of subsets of N . Our problem is then $\max_{S \in \mathcal{I}} f(S)$. We are interested in independence families induced by natural and useful constraints such as matroid constraints, knapsack constraints, related special cases, and their intersections. Throughout this paper we assume that f is given via a value oracle; that is, given a set $S \subseteq N$ the oracle returns $f(S)$. The function f could be monotone or non-monotone³; monotone functions typically allow better approximation results.

Submodular function maximization has recently attracted considerable attention in theoretical computer science. This is for a variety of reasons, including diverse applications—a prominent application field being algorithmic game

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[†]Dept. of Computer Science, Univ. of Illinois, Urbana, IL 61801, USA. Partially supported by NSF grants CCF-0728782 and CCF-1016684. E-mail: chekuri@illinois.edu.

[‡]IBM Almaden Research Center, San Jose, CA 95120, USA. E-mail: jvondrak@us.ibm.com.

[§]Dept. of Mathematics, ETH Zurich, 8092 Zurich, Switzerland, and Department of Applied Mathematics and Statistics, Johns Hopkins University, Baltimore, MD 21218, USA. E-mail: ricoz@math.ethz.ch. Supported by Swiss National Science Foundation grant PBEZP2-129524, by NSF grants CCF-1115849 and CCF-0829878, and by ONR grants N00014-12-1-0033, N00014-11-1-0053 and N00014-09-1-0326.

¹A set function $f : 2^N \rightarrow \mathbb{R}$ is submodular iff $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq N$.

²A family of sets $\mathcal{I} \subseteq 2^N$ is downward-closed if for any $A \subset B \subseteq N$, $B \in \mathcal{I}$ implies that $A \in \mathcal{I}$.

³ f is *monotone* if $f(A) \leq f(B)$ whenever $A \subseteq B$.

theory, where submodular functions are very commonly used as utility functions to describe diminishing returns—and also the recognition of interesting algorithmic and structural properties. A number of well-known problems can be seen as special cases of submodular function maximization. For example, the APX-hard Max-Cut problem can be seen as (unconstrained) maximization of the cut function $f : 2^V \rightarrow \mathbb{R}_+$ of a graph $G = (V, E)$. (Note that f here is non-monotone.) Another well-known special case of our problem is the Max- k -Cover problem, which can be viewed as $\max\{f(S) : |S| \leq k\}$ where $f(S) = |\bigcup_{j \in S} A_j|$ is the coverage function for a collection of sets $\{A_i\}$. Max- k -Cover is hard to approximate to within a factor of $(1 - 1/e + \varepsilon)$ for any fixed $\varepsilon > 0$, unless $P = NP$ [20]. Hence we focus on approximation algorithms⁴.

Classical work in submodular function maximization was based on combinatorial techniques such as the greedy algorithm and local search. We mention the work of Cornuejols, Fisher, Nemhauser and Wolsey [18, 42, 25, 41] from the late 70’s which showed a variety of approximation bounds when f is monotone submodular and \mathcal{I} is the intersection of matroid constraints. Recent algorithmic work has considerably extended and improved the classical results. Local-search methods have been identified as particularly useful, especially for non-monotone functions. Some of the recent results include the first constant factor approximation for the unconstrained submodular function maximization problem [21], and a variety of approximation results for knapsack and matroid constraints [36, 37]. The greedy algorithm has also been modified and made applicable to non-monotone functions [29].

Despite the above-mentioned results, combinatorial techniques have some limitations: (i) they have not been able to achieve optimal approximation results, except in the basic case of a single cardinality or knapsack constraint [42, 46]; (ii) they do not provide the flexibility to combine constraints of different types. A new approach which overcomes some of these obstacles and brings submodular function maximization closer to the world of polyhedral techniques is via the *multilinear relaxation*, introduced in this context in [6].

A relaxation and rounding framework based on the multilinear relaxation. In this paper we introduce a general relaxation and rounding framework for maximizing submodular functions, which builds upon, and significantly extends, previous approaches. When dealing with linear (or convex) objective functions, a standard paradigm is to design a linear or convex relaxation whose solution is then rounded via a problem-specific procedure. A difficulty faced in extending this approach to maximizing a submodular function $f : 2^N \rightarrow \mathbb{R}_+$ — which we often interpret as a function on the vertices of a $\{0, 1\}^N$ hypercube that correspond to incidence vectors — is to find a suitable extension $g : [0, 1]^N \rightarrow \mathbb{R}_+$ of f to the full hypercube. The goal is to leverage such an extension g as follows. Suppose we have a polytope $P_{\mathcal{I}} \subseteq [0, 1]^N$ that is a relaxation for $\mathcal{I} \subseteq 2^N$ in the sense that $\{\mathbf{1}_I \mid I \in \mathcal{I}\} \subset P_{\mathcal{I}}$. We want to approximately maximize the continuous problem $\max_{\mathbf{x} \in P_{\mathcal{I}}} g(\mathbf{x})$ to find a fractional solution $\mathbf{x}^* \in P_{\mathcal{I}}$ that is finally rounded to a feasible integral solution.

The best-studied extension of a submodular function is the Lovász extension [39]; however, being a convex function, it is mostly suitable for submodular function minimization problems. For maximization of submodular functions, the following *multilinear* extension was introduced in [6], inspired by the work in [1]:

$$F(\mathbf{x}) = \sum_{S \subseteq N} f(S) \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j).$$

The value $F(\mathbf{x})$ is equivalently the expected value of $f(R)$ where R is a random set obtained by picking each element i independently with probability x_i . We observe that if f is modular⁵ then F is simply a linear function. In this paper we focus on the multilinear extension. The two obvious questions that arise when trying to build a general relaxation and rounding framework based on the multilinear extension are the following. First, can we (approximately) solve the problem $\max_{\mathbf{x} \in P_{\mathcal{I}}} F(\mathbf{x})$? This question is particularly interesting due to the fact that the multilinear extension is in general not concave (nor convex). Second, can we round a fractional solution effectively?

Recent work has addressed the above questions in several ways. First, Vondrák [49] gave a continuous greedy algorithm that gives an optimal $(1 - 1/e)$ -approximation for the problem $\max_{\mathbf{x} \in P} F(\mathbf{x})$ when f is monotone submodular and P is a solvable polytope⁶. When f is non-monotone, the picture is less satisfactory. Lee et al. [36] gave a local-search based algorithm that gives a $(1/4 - \varepsilon)$ -approximation to maximize F over the polytope induced by a

⁴If f is not assumed to be non-negative, even the unconstrained problem is inapproximable since deciding whether the optimum value is positive or zero requires an exponential number of queries.

⁵A function is modular if $f(A) + f(B) = f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq N$. If f is modular then $f(A) = w_0 + \sum_{i \in A} w_i$ for some weight function $w : N \rightarrow \mathbb{R}$.

⁶We say that a polytope P is solvable if one can do efficient linear optimization over P .

fixed number of knapsack constraints. Vondrák [50] obtained a 0.309-approximation for maximizing F over a single matroid polytope, and this ratio has been recently improved to 0.325 [43]. However, no approximation algorithm was known to maximize F over a general solvable polytope P .

In terms of rounding a fractional solution \mathbf{x} , a natural strategy to preserve the value of $F(\mathbf{x})$ in expectation is to independently round each coordinate i up to 1 with probability x_i and down to 0 otherwise. However, this rounding strategy does not typically preserve the constraints imposed by \mathcal{I} . Various dependent rounding schemes have been proposed. It was shown in [6] that "pipage rounding" can be used to round solutions in the matroid polytope without losing in terms of the objective function $F(\mathbf{x})$ ([15] achieves the same via "swap-rounding"). In [33, 36, 4, 34], randomized rounding coupled with alteration was used for knapsack constraints. More recently, [15] showed concentration properties for rounding in a single matroid polytope when f is monotone, and [52] showed concentration for independent rounding even when f is non-monotone. These led to a few additional results. Despite this progress, the "integrality gap" of $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ has been so far unknown even when f is monotone and P the intersection of two matroid polytopes. (We remark that for intersections of matroids, combinatorial algorithms are known to yield good approximations [36, 37].) However, even for modular functions—i.e., classical linear optimization—combining constraints such as matroids and knapsack constraints has been difficult, and no general result was known that matched the best bounds one can get for them separately.

In summary, previous results via the multilinear relaxation were known only for rather restricted cases, both in terms of approximately maximizing the multilinear extension, and in terms of effectively rounding fractional solutions. We next describe the contributions of this paper in this context.

Our contribution at a high level: In this paper we develop a general framework for solving submodular maximization problems of the form $\max\{f(S) : S \in \mathcal{I}\}$, where $f : 2^N \rightarrow \mathbb{R}_+$ is submodular and $\mathcal{I} \subset 2^X$ is a downward-closed family of sets. Our framework consists of the following components.

- *Optimizing the multilinear relaxation:* We give the first constant factor approximation, with an additional negligible additive error, for the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ where F is the multilinear extension of any non-negative submodular function, and P is any down-monotone⁷ solvable polytope.
- *Dependent randomized rounding:* We propose a general (dependent) randomized rounding framework for modular and submodular functions under independence constraints via what we call *contention resolution schemes* (CR schemes). Rounding an approximate maximizer of the relaxation $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ via a CR scheme that is tailored to the given constraints, leads to a solution with provable approximation guarantee. A key advantage of CR schemes is the ability to easily combine CR schemes designed for different constraints into a CR scheme for the intersection of these constraints.
- *Contention resolution schemes:* We present CR schemes for a variety of packing constraints, including knapsack constraints, matroid constraints, sparse packing systems, column-restricted packing constraints, and constraints imposed by an unsplittable flow problem in paths and trees. Our CR scheme for the matroid polytope, which is provably optimal⁸, is obtained by exploiting a tight connection between CR schemes and the *correlation gap* [2] of the associated weighted rank functions. Previously, in the context of matroids, an optimal CR scheme was only known for the uniform matroid of rank 1 [22, 23].

The above ingredients can be put together to obtain a relaxation and rounding framework leading to a variety of new results that we discuss in more detail in Section 2. We summarize some of our results in Table 1.

1.1 Maximizing the multilinear extension over a general polytope

We now give a more detailed description of our technical results and the general framework. First, we give a constant factor approximation for the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$, where F is the multilinear extension of a non-monotone submodular function f and P is a down-monotone solvable polytope; the monotone case admits a $(1 - 1/e)$ -approximation [49] as we mentioned already. The condition of down-monotonicity of the polytope is necessary for

⁷A polytope $P \subseteq [0, 1]^N$ is down-monotone if for all $\mathbf{x}, \mathbf{y} \in [0, 1]^N$, $\mathbf{y} \leq \mathbf{x}$ and $\mathbf{x} \in P$ implies $\mathbf{y} \in P$.

⁸We would like to highlight that contention resolutions schemes are just one way of rounding a fractional solution. Hence, the use of an optimal contention resolution scheme does not imply that no better approximation factor can be obtained by a different procedure.

Constraint type	Linear maximization	Monotone submod. max.	Non-negative submod. max.
$O(1)$ knapsacks	$[1 - \varepsilon [8, 26]]$	$[1 - 1/e - \varepsilon [34]]$	$0.325 [0.25 [36]]$
k matroids & $\ell = O(1)$ knapsacks	$\frac{0.6}{k} [\Omega(\frac{1}{k+\ell}) [28, 29]]$	$\frac{0.38}{k} [\Omega(\frac{1}{k+\ell}) [28, 29]]$	$\frac{0.19}{k} [\Omega(\frac{1}{k+\ell}) [28, 29]]$
k -matchoid & ℓ -sparse PIP	$\Omega(\frac{1}{k+\ell})$	$\Omega(\frac{1}{k+\ell})$	$\Omega(\frac{1}{k+\ell})$
Unsplittable flow in paths and trees	$[\Omega(1) [14]]$	$\Omega(1)$	$\Omega(1)$

Table 1: Approximation ratios for different types of constraints and objective functions. Results in square brackets were previously known. The ratios in the last column for non-monotone functions are based on a 0.325 approximation for maximizing the multilinear relaxation described in the conference version of this paper [16]; there is a $\frac{1}{e} - \varepsilon \approx 0.367$ -approximation from subsequent work of [24] which results in improved bounds.

the non-monotone case; it follows from [50, 51] that no constant factor approximation is possible for the matroid base polytope which is not down-monotone.

The main algorithmic technique for non-monotone functions has been local search. Fractional local search with additional ideas has been the tool to solve the continuous problem in special cases of polytopes [36, 50, 43]. Previous fractional local search methods ([36] and [50]) improved a current solution \mathbf{x} by considering moves along a small number of coordinates of \mathbf{x} . The analysis took advantage of the combinatorial structure of the underlying constraint (knapsacks or matroids) which was sufficiently simple that swaps along a few coordinates sufficed. How do we obtain an algorithm that works for *any* polytope P ?

A new insight: Our key high-level idea is simple yet insightful. Any point $\mathbf{x} \in P$ can be written as a convex combination of the vertices of P . We view the problem of $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ as optimizing a submodular function over the ground set consisting of the (exponentially many) vertices of P (duplicated many times in the limit). From this viewpoint we obtain a new fractional local search procedure: given a current point \mathbf{x} , a local swap corresponds to removing a vertex in the convex combination of \mathbf{x} and adding a new vertex of P (with appropriate scalar multipliers). To implement this efficiently we can use linear optimization over P . (We remark that the continuous greedy algorithm for the monotone case [49] can also be interpreted with this insight.)

Our algorithms are derived using the above high-level idea. We note that when specialized to the matroid polytope or knapsack polytope which have combinatorial structure, our algorithms become simpler and in fact resemble previous algorithms. Our algorithms and proofs of approximation guarantees are in fact simpler than the previously given proofs for particular polytopes [36, 50, 43].

We present two algorithms following this idea. The first algorithm is close in spirit to the local-search algorithm of Lee et al. for knapsack constraints [36] and gives a 0.25-approximation. This algorithm, despite having a worse approximation guarantee than the second one we present, allows us to further explain and formalize the above high-level idea in a clean way. The second algorithm uses some ideas of [50] for the case of a matroid polytope and gives a 0.309-approximation with respect to the best *integer* solution in P .

We would like to mention that subsequently to the conference version of this paper, Feldman et al. [24] presented an improved algorithm to maximize the multilinear extension, leading to an $(e^{-1} - \epsilon) \approx 0.367$ -approximation with respect to the best integer solution. Their algorithm is an adaptation of the continuous greedy algorithm [49]. The conference version of this paper [16] contained a third and much more involved algorithm (generalizing the simulated annealing approach of [43]) that gives a 0.325-approximation, again with respect to the best integer solution in P . For conciseness, and in view of the recent results in [24], we do not include this third algorithm in this paper, and concentrate on the first two algorithms mentioned in the preceding paragraph, which allow us to demonstrate the main new algorithmic insights. We summarize our results in the following theorem.

Theorem 1.1. *Let f be a nonnegative submodular function and $P \subseteq \mathbb{R}^n$ be a solvable down-monotone polytope satisfying that there is a $\lambda \in \Omega(\frac{1}{\text{poly}(n)})$ such that for each coordinate $i \in [n]$, $\lambda \cdot \mathbf{e}_i \in P$. then there is a $(0.25 - o(1))$ -approximation algorithm for the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ where F is the multilinear extension of f . There is also an algorithm for this problem which returns a solution $\mathbf{y} \in P$ of value $F(\mathbf{y}) \geq 0.309 \cdot \max\{F(\mathbf{x}) : \mathbf{x} \in P \cap \{0, 1\}^N\}$.*

We remark that a known limit on the approximability of $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ is an information theoretic hardness of 0.478-approximation in the value oracle model, even in the special case of a matroid polytope [43].

1.2 Contention resolution schemes

We show that a certain natural class of rounding schemes that we call *contention resolution schemes* (CR schemes) provides a useful and general framework for rounding fractional solutions under submodular objective functions. For a ground set N , let $P_{\mathcal{I}}$ be a convex relaxation of the constraints imposed by $\mathcal{I} \subseteq 2^N$, and let $\mathbf{x} \in P_{\mathcal{I}}$. From the definition of F , a natural strategy to round a point \mathbf{x} is to independently round the coordinates; however, this is unlikely to preserve the constraints imposed by \mathcal{I} . Let $R(\mathbf{x}) \subseteq N$ be a random set obtained by including each element $i \in N$ independently with probability x_i . The set $R(\mathbf{x})$ is not necessarily feasible. We would like to remove (randomly) some elements from $R(\mathbf{x})$, so that we obtain a feasible set $I \subseteq R(\mathbf{x})$. The property we would like to achieve is that every element i appears in I with probability at least cx_i for some parameter $c > 0$. We call such a scheme “ c -balanced contention resolution” for $P_{\mathcal{I}}$. We stress that a c -balanced CR scheme needs to work for all $\mathbf{x} \in P_{\mathcal{I}}$. However, often, stronger schemes—i.e. with larger values for c —can be obtained if they only need to work for all points in a scaled-down version $bP_{\mathcal{I}} = \{b \cdot \mathbf{x} \mid \mathbf{x} \in P_{\mathcal{I}}\}$ of $P_{\mathcal{I}}$, where $b \in [0, 1]$. Such schemes, which we call (b, c) -balanced schemes, will prove to be useful when combining CR schemes for different constraints as we will discuss in Section 1.3. Below is a formal definition of CR schemes. Let $\text{support}(\mathbf{x}) = \{i \in N \mid x_i > 0\}$.

Definition 1.2. Let $b, c \in [0, 1]$. A (b, c) -balanced CR scheme π for $P_{\mathcal{I}}$ is a procedure that for every $\mathbf{x} \in bP_{\mathcal{I}}$ and $A \subseteq N$, returns a random set $\pi_{\mathbf{x}}(A) \subseteq A \cap \text{support}(\mathbf{x})$ and satisfies the following properties:

- (i) $\pi_{\mathbf{x}}(A) \in \mathcal{I}$ with probability 1 $\forall A \subseteq N, \mathbf{x} \in bP_{\mathcal{I}}$, and
- (ii) for all $i \in \text{support}(\mathbf{x})$, $\Pr[i \in \pi_{\mathbf{x}}(R(\mathbf{x})) \mid i \in R(\mathbf{x})] \geq c \quad \forall \mathbf{x} \in bP_{\mathcal{I}}$.

The scheme is said to be *monotone* if $\Pr[i \in \pi_{\mathbf{x}}(A_1)] \geq \Pr[i \in \pi_{\mathbf{x}}(A_2)]$ whenever $i \in A_1 \subseteq A_2$. A $(1, c)$ -balanced CR scheme is also called a c -balanced CR scheme. The scheme is *deterministic* if π is a deterministic algorithm (hence $\pi_{\mathbf{x}}(A)$ is a single set instead of a distribution). It is *oblivious* if π is deterministic and $\pi_{\mathbf{x}}(A) = \pi_{\mathbf{y}}(A)$ for all \mathbf{x}, \mathbf{y} and A , that is, the output is independent of \mathbf{x} and only depends on A . The scheme is *efficiently implementable* if π is a polynomial-time algorithm that given \mathbf{x}, A outputs $\pi_{\mathbf{x}}(A)$.

We emphasize that a CR scheme is defined with respect to a specific polyhedral relaxation $P_{\mathcal{I}}$ of \mathcal{I} . Note that on the left-hand side of condition (ii) for a CR scheme, the probability is with respect to two random sources: first the set $R(\mathbf{x})$ is a random set, and second, the procedure $\pi_{\mathbf{x}}$ is typically randomized. We note that a (b, c) -balanced CR scheme π can easily be transformed into a bc -balanced CR scheme; details are given in Section 4.

The theorem below highlights the utility of CR schemes; when rounding via monotone contention resolution schemes, one can claim an expectation bound for submodular functions. A similar theorem was shown in [4] for monotone functions. We state and prove ours in a form suitable for our context.

Theorem 1.3. Let $b, c \in [0, 1]$, and let $f : 2^N \rightarrow \mathbb{R}_+$ be a non-negative submodular function with multilinear relaxation F , and $\mathbf{x} \in b \cdot P_{\mathcal{I}}$, where $P_{\mathcal{I}}$ is a convex relaxation for $\mathcal{I} \subseteq 2^N$. Furthermore, let π be a monotone (b, c) -balanced CR scheme for $P_{\mathcal{I}}$, and let $I = \pi_{\mathbf{x}}(R(\mathbf{x}))$. If f is monotone then

$$\mathbf{E}[f(I)] \geq cF(\mathbf{x}).$$

Furthermore, there is a function $\eta_f : 2^N \rightarrow 2^N$ that depends on f and can be evaluated in linear time, such that even for f non-monotone

$$\mathbf{E}[f(\eta_f(I))] \geq cF(\mathbf{x}).$$

As we will see in Section 4, the function η_f can be chosen to always return a subset of its argument. We therefore call it a *pruning operation*.

We observe that several previous rounding procedures for packing (and also covering) problems rely on the well-known technique of *alteration* of a set obtained via independent rounding and are examples of CR schemes (see [45, 5, 9, 14, 4]). However, these schemes are typically oblivious in that they do not depend on \mathbf{x} itself (other than in picking the random set R), and the alteration is deterministic. Our definition is inspired by the “fair contention resolution scheme” in [22, 23] which considered the special case of contention for a single item. The dependence on \mathbf{x} as well as randomization is necessary (even in this case) if we want to obtain an optimal scheme. One key question

to consider is whether some given down-monotone polytope $P_{\mathcal{I}}$ admits a “good” (b, c) -balanced CR scheme, which corresponds to having values of b and c that are as close to 1 as possible.

One natural way to apply a (b, c) -balanced CR scheme to a point $\hat{x} \in P_{\mathcal{I}}$ that approximately maximizes F is as follows. In a first step we scale down \hat{x} to obtain $x = b \cdot \hat{x}$. By non-negativity and concavity of F along non-negative directions one obtains $F(x) \geq b \cdot F(\hat{x})$. Applying a (b, c) -balanced CR scheme π to x leads to a set $I = \pi(R(x))$ which, according to Theorem 1.3, satisfies $f(I) \geq cF(x) \geq cbF(\hat{x})$. This also highlights the motivation why we want to have b and c as close to 1 as possible.

As we will show, many natural constraint systems admit good (b, c) -balanced CR schemes, including matroid constraints, knapsack constraints, and a variety of packing integer programs. In particular, to deal with the rather general class of matroid constraints, we exploit a close connection between the existence of CR schemes and a recently introduced concept, called *correlation gap* [53].

Contention resolution via correlation gap and an optimal scheme for matroids: Until recently there was no contention resolution scheme for the matroid polytope; an optimal $(b, \frac{1-e^{-b}}{b})$ -balanced scheme was previously known for the very special case of the uniform matroid of rank one [22, 23]. We note that the recent work of Chawla et al. [11, 12] implicitly contains a $(b, 1 - b)$ -balanced deterministic scheme for matroids; their motivation for considering this notion was mechanism design. In this paper we develop an optimal scheme for an arbitrary matroid⁹.

Theorem 1.4. *There is an optimal $(b, \frac{1-e^{-b}}{b})$ -balanced contention resolution scheme for any matroid polytope. Moreover the scheme is monotone and efficiently implementable.*

The main idea in proving the preceding theorem is consider a randomized CR scheme and view it abstractly as a convex combination of deterministic CR schemes. This allows, via LP duality, to show that the best contention resolution scheme for a constraint system is related to the notion of correlation gap for weighted rank functions of the underlying constraint. We reiterate that the scheme depends on the fractional solution \mathbf{x} that we wish to round; the alteration of the random set $R(\mathbf{x})$ is itself a randomized procedure that is tailored to \mathbf{x} , and is found by solving a linear program. We are inspired to make the general connection to correlation gap due to the recent work of Yan [53]; he applied a similar idea in the context of greedy posted-price ordering schemes for Bayesian mechanism design, improving the bounds of [11, 12].

1.3 A framework for rounding via contention resolution schemes

We now describe our framework for the problem $\max_{S \in \mathcal{I}} f(S)$. The framework assumes the following: (i) there is a polynomial-time value oracle for f , (ii) there is a solvable down-monotone polytope $P_{\mathcal{I}}$ that contains the set $\{\mathbf{1}_S \mid S \in \mathcal{I}\}$, and (iii) there is a monotone c -balanced contention resolution scheme π for $P_{\mathcal{I}}$. Then we have the following simple algorithm:

1. Using an approximation algorithm, obtain in polynomial time a point $\mathbf{x}^* \in P_{\mathcal{I}}$ such that

$$F(\mathbf{x}^*) \geq \alpha \cdot \max\{F(\mathbf{x}) \mid \mathbf{x} \in P_{\mathcal{I}} \cap \{0, 1\}^N\} \geq \alpha \cdot \max_{S \in \mathcal{I}} f(S).$$

2. Round the point \mathbf{x}^* using the CR scheme π to obtain $I = \pi_{\mathbf{x}^*}(R(\mathbf{x}^*))$, and return its pruned version $\eta_f(I)$.

Theorem 1.5. *The preceding framework gives a randomized (αc) -approximation algorithm for $\max_{S \in \mathcal{I}} f(S)$, whenever f is non-negative submodular, α is the approximation ratio for $\max\{F(\mathbf{x}) \mid \mathbf{x} \in P_{\mathcal{I}} \cap \{0, 1\}^N\}$ and $P_{\mathcal{I}}$ admits a monotone c -balanced CR scheme. If f is monotone then the pruning step is not needed. If f is modular then the ratio is c and the CR scheme is not even constrained to be monotone.*

Proof. We have $F(\mathbf{x}^*) \geq \alpha \text{OPT}$ with $\text{OPT} = \max_{S \in \mathcal{I}} f(S)$. Theorem 1.3 shows that $\mathbf{E}[f(\eta_f(I))] \geq cF(\mathbf{x}^*)$, hence $\mathbf{E}[f(\eta_f(I))] \geq \alpha c \text{OPT}$. If f is monotone, the pruning step is not required by Theorem 1.3.

For modular f , $F(\mathbf{x})$ is a linear function, and hence $\alpha = 1$ can be obtained by linear programming. Moreover, if $F(\mathbf{x})$ is a linear function, then by linearity of expectation, $\mathbf{E}[f(I)] \geq cF(\mathbf{x}^*)$ without any monotonicity assumption on the scheme. \square

⁹We also describe the $(b, 1 - b)$ scheme in Section 4.4 for completeness. This scheme is simpler and computationally advantageous when compared to the optimal scheme.

For non-monotone submodular functions, Theorem 1.1 gives $\alpha = 0.309$; the currently best known approximation is $(\frac{1}{e} - \varepsilon) \simeq 0.367$ due to [24]. For monotone submodular functions an optimal bound of $\alpha = 1 - \frac{1}{e}$ is given in [49].

Combining schemes for different constraints: We are particularly interested in the case when $\mathcal{I} = \cap_{i=1}^h \mathcal{I}_i$ is the intersection of several different independence systems on N ; each system corresponds to a different set of constraints that we would like to impose. Assuming that we can apply the above framework to each \mathcal{I}_i separately, we can obtain an algorithm for \mathcal{I} as follows.

Lemma 1.6. *Let $\mathcal{I} = \cap_{i=1}^h \mathcal{I}_i$ and $P_{\mathcal{I}} = \cap_i P_{\mathcal{I}_i}$. Suppose each $P_{\mathcal{I}_i}$ has a monotone (b, c_i) -balanced CR scheme. Then $P_{\mathcal{I}}$ has a monotone $(b, \prod_i c_i)$ -balanced CR scheme. In the special case that each element of N participates in at most k constraints and $c_i = c$ for all i then $P_{\mathcal{I}}$ has a monotone (b, c^k) -balanced CR scheme. Moreover, if the scheme for each $P_{\mathcal{I}_i}$ is implementable in polynomial time then the combined scheme for $P_{\mathcal{I}}$ can be implemented in polynomial time.*

Therefore, we can proceed as follows. Let $P_{\mathcal{I}_i}$ be a polytope that is the relaxation for \mathcal{I}_i . In other words $\{\mathbf{1}_S : S \in \mathcal{I}_i\}$ is contained in $P_{\mathcal{I}_i}$. Let $P_{\mathcal{I}} = \cap_i P_{\mathcal{I}_i}$. It follows that $\{\mathbf{1}_S : S \in \mathcal{I}\}$ is contained in $P_{\mathcal{I}}$ and also that there is a polynomial-time separation oracle for $P_{\mathcal{I}}$ if there is one for each $P_{\mathcal{I}_i}$. Now suppose there is a monotone (b, c_i) -balanced contention resolution scheme for $P_{\mathcal{I}_i}$ for some common choice of b . It follows from Lemma 1.6 that $P_{\mathcal{I}}$ has a monotone $(b, \prod_i c_i)$ -balanced contention resolution scheme, which can be transformed into a $(b \prod_i c_i)$ -balanced scheme for $P_{\mathcal{I}}$. We can then apply Theorem 1.5 to obtain a randomized $(\alpha b \prod_i c_i)$ -approximation for $\max_{S \in \mathcal{I}} f(S)$ where α depends on whether f is modular, monotone submodular or non-monotone submodular.

In this paper we focus on the framework with a small list of high-level applications. We have not attempted to optimize for the best possible approximation for special cases. We add two remarks that are useful in augmenting the framework.

Remark 1.7. *Whenever the rounding step of our framework is performed by a CR scheme that was obtained from a (b, c) -balanced CR scheme—in particular in the context mentioned above when combining CR schemes for different constraints—we can often strengthen the procedure as follows. Instead of approximately solving $\max_{\mathbf{x} \in P_{\mathcal{I}}} F(\mathbf{x})$, we can approximately solve $\max_{\mathbf{y} \in bP_{\mathcal{I}}} F(\mathbf{y})$ to obtain \mathbf{y}^* , and then directly apply the (b, c) -balanced scheme to \mathbf{y}^* , without transforming it first to a bc -balanced scheme. This may be advantageous if the problem $\max_{\mathbf{y} \in bP_{\mathcal{I}}} F(\mathbf{y})$ admits a direct approximation better than one obtained by scaling from $\max_{\mathbf{y} \in P_{\mathcal{I}}} F(\mathbf{y})$. A useful fact here is that the continuous greedy algorithm for monotone submodular functions [50, 7] finds for every $b \in [0, 1]$ a point $\mathbf{y}^* \in bP_{\mathcal{I}}$ such that $F(\mathbf{y}^*) \geq (1 - e^{-b}) \max_{\mathbf{x} \in P_{\mathcal{I}}} F(\mathbf{x})$. This is indeed a stronger guarantee than the one obtained by first applying the continuous greedy to $P_{\mathcal{I}}$ to obtain \mathbf{x}^* , and then used the scaled-down version $b\mathbf{x}^*$, which leads to a guarantee of only $F(b\mathbf{x}^*) \geq bF(\mathbf{x}^*) \geq b(1 - e^{-1}) \max_{\mathbf{x} \in P_{\mathcal{I}}} F(\mathbf{x})$.*

Remark 1.8. *A non-negative submodular set function f is also subadditive, that is, $f(A) + f(B) \geq f(A \cup B)$. In some settings when considering the problem $\max_{S \in \mathcal{I}} f(S)$, it may be advantageous to partition the given ground set N into N_1, \dots, N_h , separately solve the problem on each N_i , and then return the best of these solutions. This loses a factor of h in the approximation but one may be able to obtain a good CR scheme for each N_i separately while it may not be straightforward to obtain one for the entire set N .*

An application of the technique mentioned in Remark 1.8 can be found in Section 4.8, where we use it in the context of column-restricted packing constraints.

Organization: The rest of the paper is divided into three parts. Some illustrative applications of our framework are discussed in Section 2. Constant factor approximation algorithms for maximizing F over a solvable polytope are described in Section 3. Section 4 discusses the construction of CR schemes. This include a discussion of the connection between contention resolution schemes and correlation gap and its use in deriving optimal schemes for matroids. Furthermore, in the same section, we present CR schemes for knapsack constraints, sparse packing systems, and UFP in paths and trees.

2 Applications

In this section we briefly outline some concrete results that can be obtained via our framework. The meta-problem we are interested in solving is $\max_{S \in \mathcal{I}} f(S)$ where \mathcal{I} is a downward-closed family over the given ground set N and f is a non-negative submodular set function over N . Many interesting problems can be cast as special cases depending on the choice of N , \mathcal{I} and f . In order to apply the framework and obtain a polynomial-time approximation algorithm, we need a solvable relaxation $P_{\mathcal{I}}$ and a corresponding (b, c) -balanced CR scheme. Note that the framework is essentially indifferent to f as long as we have a polynomial-time value oracle for it. We therefore focus on some broad classes of constraints and corresponding natural polyhedral relaxations, and discuss CR schemes that can be obtained for them. These schemes are formally described in Section 4.

Matroids and matchoids: Let $\mathcal{M} = (N, \mathcal{I})$ be a matroid constraint on N . A natural candidate for $P_{\mathcal{I}}$ is the integral matroid polytope $\{x \in [0, 1]^n \mid x(S) \leq r(S), S \subseteq N\}$ where $r : 2^N \rightarrow \mathbb{Z}_+$ is the rank function of \mathcal{M} . We develop an optimal $(1 - 1/e)$ -balanced CR scheme for the matroid polytope. More generally, for any $b \in (0, 1]$ we design a $(b, \frac{1-e^{-b}}{b})$ -balanced CR scheme, which lends itself well to combinations with other constraints. The CR scheme for the matroid polytope extends via Lemma 1.6 to the case when \mathcal{I} is induced by the intersection of k matroid constraints on N . A more general result is obtained by considering k -uniform matchoids, a common generalization of k -set packing and intersection of k matroids [38], defined as follows. Let $G = (V, N)$ be a k -uniform hypergraph; we associate the edges of the hypergraph with our ground set N . For each $v \in V$, there is a matroid $\mathcal{M}_v = (N_v, \mathcal{I}_v)$ over N_v , set of hyperedges in N that contain v . This induces an independence family \mathcal{I} on N where $\mathcal{I} = \{S \subseteq N \mid S \cap N_v \in \mathcal{I}_v, v \in V\}$. k -uniform matchoids generalize the intersection of k matroids in that they allow many matroids in the intersection as long as a given element of the ground set participates in at most k of them. A natural solvable relaxation for \mathcal{I} is the intersection of the matroid polytopes at each v . Via the CR scheme for the single matroid and Lemma 1.6 we obtain a $(b, (\frac{1-e^{-b}}{b})^k)$ -balanced CR scheme for any $b \in (0, 1]$ for k -uniform matchoids. The choice of $b = \frac{2}{k+1}$ gives a $\frac{2}{e^{k+1}}$ -balanced CR scheme for every k -uniform matchoid.

Knapsack / linear packing constraints: Let $N = \{1, 2, \dots, n\}$. Given a non-negative $m \times n$ matrix A and non-negative vector \mathbf{b} , let $\mathcal{I} = \{S \mid A\mathbf{1}_S \leq \mathbf{b}\}$ where $\mathbf{1}_S$ is the indicator vector of set $S \subseteq N$. It is easy to see that \mathcal{I} is an independence family. A natural LP relaxation for the problem is $P_{\mathcal{I}} = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, x \in [0, 1]^n\}$. The *width* of the system of inequalities is defined as $W = \lfloor \min_{i,j} b_i / A_{i,j} \rfloor$. Some special cases of interest are (i) A is a $\{0, 1\}$ -matrix, (ii) A is column-restricted, that is, all non-zero entries in each column are the same and (iii) A is k -column sparse, that is at most k non-zero entries in each column. Several combinatorial problems can be captured by these, such as matchings and independent sets in graphs and hypergraphs, knapsack and its variants, and maximum throughput routing problems. However, the maximum independent set problem in graphs, which is a special case as mentioned, does not allow a $n^{1-\varepsilon}$ -approximation for any fixed $\varepsilon > 0$, unless P=NP [30]. Therefore attention has focused on restricting A in various ways and obtaining upper bounds on the integrality gap of the relaxation $P_{\mathcal{I}}$ when the objective function is linear. Several of these results are based on randomized rounding of a fractional solution and one can interpret the rounding algorithms as CR schemes. We consider a few such results below.

- For a constant number of knapsack constraints ($m = O(1)$), by guessing and enumeration tricks, one can “effectively” get a $(1 - \varepsilon, 1 - \varepsilon)$ -balanced CR scheme for any fixed $\varepsilon > 0$.
- When A is k -column sparse, there is a $(b, 1 - 2kb)$ -balanced CR scheme. If A has in addition width $W \geq 2$, there is a $(b, 1 - k(2eb)^{W-1})$ CR scheme for any $b \in (0, 1)$. These results follow from [4].
- When A is a $\{0, 1\}$ -matrix induced by the problem of routing unit-demand paths in a capacitated path or tree, there is a $(b, 1 - O(b))$ CR scheme implicit in [5, 9, 14]. This can be extended to the unsplittable flow problem (UFP) in capacitated paths and trees via grouping and scaling techniques [31, 14, 13].

Section 4 has formal details of the claimed CR schemes. There are other rounding schemes in the literature for packing problems, typically developed for linear functions, that can be reinterpreted as CR schemes. Our framework can then be used to obtain algorithms for non-negative submodular set functions. See [10] for a recent and illuminating example.

Approximation algorithms. The CR schemes mentioned above when instantiated with suitable parameters and plugged into our general framework yield several new randomized polynomial-time approximation algorithms for

problems of the form $\max_{S \in \mathcal{I}} f(S)$, where f is non-negative submodular. We remark that these results are for somewhat abstract problems and one can obtain more concrete results by specializing them and improving the constants. We have not attempted to do so in this paper.

- If \mathcal{I} is the intersection of a fixed number of knapsack constraints, we achieve a 0.309-approximation, improving the $(0.2 - \varepsilon)$ -approximation from [36] and a recent $(0.25 - \varepsilon)$ -approximation [34]. This is obtained via the $(1 - \varepsilon, 1 - \varepsilon)$ -balanced CR scheme for a fixed number of knapsack constraints.
- If \mathcal{I} is the intersection of a k -uniform matchoid and ℓ knapsack constraints with ℓ a fixed constant, we obtain an $\Omega(\frac{1}{k})$ -approximation (constant independent of ℓ), which improves the bound of $\Omega(\frac{1}{k+\ell})$ from [28]. We remark that this is a new result even for linear objective functions. We obtain this by choosing $b = \Omega(1/k)$ and using the $(b, (\frac{1-e^{-b}}{b})^k)$ -balanced CR scheme for k -uniform matchoids and the $(1 - \varepsilon, 1 - \varepsilon)$ -balanced CR scheme for a fixed number of knapsack constraints (this requires a separate preprocessing step).
- If \mathcal{I} is the intersection of a k -uniform matchoid and an ℓ -sparse knapsack constraint system of width W , we give an $\Omega(\frac{1}{k+\ell^{1/W}})$ -approximation, improving the $\Omega(\frac{1}{k\ell})$ approximation from [28]. This follows by combining the CR schemes for k -uniform matchoid and ℓ -column sparse packing constraints with a choice of $b = \Omega(\frac{1}{k+\ell^{1/W}})$.
- We obtain a constant factor approximation for maximizing a non-negative submodular function of routed requests in a capacitated path or tree. Previously an $O(1)$ approximation was known for linear functions [5, 9, 14, 13].

3 Solving the multilinear relaxation for non-negative submodular functions

In this section, we address the question of solving the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ where F is the multilinear extension of a submodular function. As we already mentioned, due to [49, 7], there is a $(1 - 1/e)$ -approximation for the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ whenever F is the multilinear extension of a monotone submodular function and P is any solvable polytope. Here, we consider the maximization of a possibly *non-monotone submodular function* over a down-monotone solvable polytope. We assume in the following that $P \subseteq [0, 1]^N$ is a down-monotone solvable polytope and $F : [0, 1]^N \rightarrow \mathbb{R}_+$ is the multilinear extension of a submodular function. We present two algorithms for this problem. As we noted in the introduction, there is no constant-factor approximation for maximizing non-monotone submodular functions over general—i.e., not necessarily down-monotone—solvable polytopes [50]. The approximation that can be achieved for matroid base polytopes is proportional to $1 - 1/\nu$ where ν is the fractional packing number of bases (see [50]), and in fact this trade-off generalizes to arbitrary solvable polytopes; we discuss this in Appendix A.

3.1 Continuous local-search

Here we present our first algorithm for the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$. We remark that in the special case of multiple knapsack constraints, this algorithm is equivalent to the algorithm of [36].

First we consider a natural local-search algorithm that tries to find a local optimum for F in the polytope P . For a continuous function g defined over a convex set $C \subseteq \mathbb{R}^n$, a point $\mathbf{x} \in C$ is a local optimum (in particular, a maximum), if $g(\mathbf{x}) \geq g(\mathbf{x}')$ for all $\mathbf{x}' \in C$ in a neighborhood of \mathbf{x} . If g is differentiable over C , a first-order necessary condition for \mathbf{x} to be a local maximum is that $(\mathbf{y} - \mathbf{x}) \cdot \nabla g(\mathbf{x}) \leq 0$ for all $\mathbf{y} \in C$. If g is in addition a concave function then this is in fact sufficient for \mathbf{x} to be a global maximum. However, in general the first-order necessary condition is not sufficient to guarantee even a local optimum. Although sufficient conditions based on second-order partial derivatives exist, it is non-trivial to find a local optimum or to certify that a given point \mathbf{x} is a local optimum. Our algorithms and analysis rely only on finding a point which satisfies (approximately) the first-order necessary condition. Hence, this point is not necessarily a local optimum in the classical sense. Nevertheless, for notational convenience we refer to any such point as a local optimum (sometimes such a point is referred to as a constrained critical point). A simple high-level procedure to find such a local optimum for $F(\mathbf{x})$ in P —which does not consider implementability—is the following. We will subsequently discuss how to obtain an efficient version of this high-level approach that returns an approximate local optimum.

Algorithm 3.1. *Continuous local search:* Initialize $\mathbf{x} := 0$. As long as there is $\mathbf{y} \in P$ such that $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) > 0$, move \mathbf{x} continuously in the direction $\mathbf{y} - \mathbf{x}$. If there is no such $\mathbf{y} \in P$, return \mathbf{x} .

This algorithm is similar to gradient descent (or rather ascent), and without considering precision and convergence issues, it would be equivalent to it. The importance of the particular formulation that we stated here will become more clear when we discretize the algorithm, in order to argue that it terminates in polynomial time and achieves a solution with suitable properties.

The objective function F is not concave; however, submodularity implies that along any non-negative direction F is concave (see [49, 7]). This leads to the following basic lemma and its corollary about local optima that we rely on in the analysis of our algorithms. In the following $\mathbf{x} \vee \mathbf{y}$ denotes the vector obtained by taking the coordinate-wise maximum of the vectors \mathbf{x} and \mathbf{y} ; and $\mathbf{x} \wedge \mathbf{y}$ denotes the vector obtained by taking the coordinate-wise minimum.

Lemma 3.2. *For any two points $\mathbf{x}, \mathbf{y} \in [0, 1]^N$ and the multilinear extension $F : [0, 1]^N \rightarrow \mathbb{R}$ of a submodular function,*

$$(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \geq F(\mathbf{x} \vee \mathbf{y}) + F(\mathbf{x} \wedge \mathbf{y}) - 2F(\mathbf{x}).$$

Proof. By submodularity, F is concave along any line with a nonnegative direction vector, such as $(\mathbf{x} \vee \mathbf{y}) - \mathbf{x} \geq 0$. Therefore,

$$\begin{aligned} F(\mathbf{x} \vee \mathbf{y}) - F(\mathbf{x}) &\leq ((\mathbf{x} \vee \mathbf{y}) - \mathbf{x}) \cdot \nabla F(\mathbf{x}), \text{ and similarly} \\ F(\mathbf{x} \wedge \mathbf{y}) - F(\mathbf{x}) &\leq ((\mathbf{x} \wedge \mathbf{y}) - \mathbf{x}) \cdot \nabla F(\mathbf{x}), \end{aligned}$$

because of the concavity of F along direction $(\mathbf{x} \wedge \mathbf{y}) - \mathbf{x} \leq 0$. Adding up these two inequalities, we get $F(\mathbf{x} \vee \mathbf{y}) + F(\mathbf{x} \wedge \mathbf{y}) - 2F(\mathbf{x}) \leq ((\mathbf{x} \vee \mathbf{y}) + (\mathbf{x} \wedge \mathbf{y}) - 2\mathbf{x}) \cdot \nabla F(\mathbf{x})$. It remains to observe that $(\mathbf{x} \vee \mathbf{y}) + (\mathbf{x} \wedge \mathbf{y}) = \mathbf{x} + \mathbf{y}$, which proves the lemma. \square

Corollary 3.3. *If \mathbf{x} is a local optimum in P , i.e. $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \leq 0$ for all $\mathbf{y} \in P$, then $2F(\mathbf{x}) \geq F(\mathbf{x} \vee \mathbf{y}) + F(\mathbf{x} \wedge \mathbf{y})$ for any $\mathbf{y} \in P$.*

3.2 Discretized local search

What follows is a discretization of Algorithm 3.1, which is the one we actually use in our framework. Let $M = \max\{f(i), f(N - i) : i \in N\}$. Notice that M is an upper bound on the maximum absolute marginal value of any element, i.e., $M \geq \max_{S,i} |f_S(i)| = \max\{f(i) - f(\emptyset), f(N - i) - f(N) : i \in N\}$. By subadditivity, we have $|f(S)| \leq Mn$ for all S . It can be also verified easily that $|\frac{\partial F}{\partial x_i}| \leq M$ and $|\frac{\partial^2 F}{\partial x_i \partial x_j}| \leq 2M$ for all i, j (see [50]). We pick a parameter $q = n^a$ for some sufficiently large constant $a > 3$ and maintain a convex combination $\mathbf{x} = \frac{1}{q} \sum_{i=1}^q \mathbf{v}_i$, where \mathbf{v}_i are certain points in P (without loss of generality vertices, with possible repetition). Each discrete step corresponds to replacing a vector in the convex combination by another. Instead of the gradient $\nabla F(\mathbf{x})$, we use an estimate of its coordinates $\frac{\partial F}{\partial x_i}$ by random sampling. We use the following lemma to control the errors in our estimates.

Lemma 3.4. *Let $\tilde{F}(\mathbf{x}) = \frac{1}{H} \sum_{h=1}^H f(R_h)$ where R_h is a random set sampled independently with probabilities x_i . Let $H = n^{2a+1}$, $\delta = M/n^{a-1}$ and $M = \max\{f(i), f(N - i) : i \in N\}$. Then the probability that $|\tilde{F}(\mathbf{x}) - F(\mathbf{x})| > \delta$ is at most $2e^{-n/8}$.*

Proof. Let us define $X_h = \frac{1}{2Mn}(f(R_h) - F(\mathbf{x}))$, a random variable bounded by 1 in absolute value. By definition, $\mathbf{E}[X_h] = 0$. By the Chernoff bound, $\Pr[\sum_{h=1}^H X_h > t] < 2e^{-t^2/2H}$ (see Theorem A.1.16 in [3]). We set $H = n^{2a+1}$ and $t = \frac{1}{2}n^{a+1}$, and obtain $\Pr[|\tilde{F}(\mathbf{x}) - F(\mathbf{x})| > M/n^{a-1}] = \Pr[\sum_{h=1}^H X_h > \frac{1}{2}n^{a+1}] < 2e^{-n/8}$. \square

Given estimates of $F(\mathbf{x})$, we can also estimate $\frac{\partial F}{\partial x_i} = F(\mathbf{x} \vee \mathbf{e}_i) - F((\mathbf{x} \vee \mathbf{e}_i) - \mathbf{e}_i) = \mathbf{E}[f(R + i) - f(R - i)]$. The above implies the following bound.

Corollary 3.5. *Let $\delta = M/n^{a-1}$. If the total number of evaluations of F and $\frac{\partial F}{\partial x_i}$ is bounded by n^b and each estimate is computed independently using n^{2a+1} samples, then with probability at least $1 - O(n^b e^{-n/8})$ all the estimates are within $\pm\delta$ additive error.*

The algorithm works as follows. The input to the algorithm is a submodular function f given by a value oracle, and a polytope P given by a separation oracle.

Algorithm 3.6. *Fractional local search.* Let $q = n^a, \delta = M/n^{a-1}$. Let $\mathbf{x} := \frac{1}{q} \sum_{i=1}^q \mathbf{v}_i$, and initialize $\mathbf{v}_i = 0$ for all i . Use estimates $\tilde{\nabla}F(\mathbf{x})$ of $\nabla F(\mathbf{x})$ within $\pm\delta$ in each coordinate. As long as there is $\mathbf{y} \in P$ such that $(\mathbf{y} - \mathbf{x}) \cdot \tilde{\nabla}F(\mathbf{x}) > 4\delta n$ (which can be found by linear programming), we modify $\mathbf{x} := \frac{1}{q} \sum_{i=1}^q \mathbf{v}_i$ by replacing one of the vectors \mathbf{v}_i in the linear combination by \mathbf{y} , so that we maximize $F(\mathbf{x})$. If there is no such $\mathbf{y} \in P$, return \mathbf{x} .

Lemma 3.7. *Algorithm 3.6 terminates in polynomial time with high probability.*

Proof. We show that if all estimates of ∇F computed during the algorithm are within $\pm\delta$ in each coordinate—which happens with high probability—then the algorithm terminates in polynomial time. This implies the lemma since with high probability, we have that a polynomial number of estimates of ∇F are indeed all within $\pm\delta$ in each coordinate. Hence, we assume in the following that all estimates $\tilde{\nabla}F$ of ∇F are within $\pm\delta$.

In each step, the algorithm continues only if it finds $\mathbf{y} \in P$ such that $(\mathbf{y} - \mathbf{x}) \cdot \tilde{\nabla}F(\mathbf{x}) \geq 4\delta n$. Since $\tilde{\nabla}F$ approximates ∇F within $\pm\delta$ in each coordinate, this means that $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \geq 3\delta n$. Denote by \mathbf{x}' a random vector that is obtained by replacing a random vector \mathbf{v}_i by \mathbf{y} , in the linear combination $\mathbf{x} = \frac{1}{q} \sum_{i=1}^q \mathbf{v}_i$. The expected effect of this change is

$$\mathbf{E}[F(\mathbf{x}') - F(\mathbf{x})] = \frac{1}{q} \sum_{i=1}^q \left(F\left(\mathbf{x} + \frac{1}{q}(\mathbf{y} - \mathbf{v}_i)\right) - F(\mathbf{x}) \right) = \frac{1}{q^2} \sum_{i=1}^q (\mathbf{y} - \mathbf{v}_i) \cdot \nabla F(\tilde{\mathbf{x}}_i)$$

where $\tilde{\mathbf{x}}_i$ is some point on the line between \mathbf{x} and $\mathbf{x} + \frac{1}{q}(\mathbf{y} - \mathbf{v}_i)$, following from the mean-value theorem. Since $q = n^a$ and the second partial derivatives of F are bounded by $2M$, we get by standard bounds that $\|\nabla F(\tilde{\mathbf{x}}_i) - \nabla F(\mathbf{x})\|_1 \leq \frac{n^2}{q} \cdot 2M = \frac{2M}{n^{a-2}} = 2\delta n$. Using also the fact that $\mathbf{y} - \mathbf{v}_i \in [-1, 1]^n$,

$$\mathbf{E}[F(\mathbf{x}') - F(\mathbf{x})] \geq \frac{1}{q^2} \sum_{i=1}^q ((\mathbf{y} - \mathbf{v}_i) \cdot \nabla F(\mathbf{x}) - 2\delta n) = \frac{1}{q} ((\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) - 2\delta n) \geq \frac{1}{q} \cdot \delta n$$

using the fact that $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \geq 3\delta n$. Therefore, if we exchange \mathbf{y} for the vertex \mathbf{v}_i that maximizes our gain, we gain at least $F(\mathbf{x}') - F(\mathbf{x}) \geq \frac{1}{q} \delta n = \frac{M}{n^{2a-2}}$. Also we have the trivial bound $\max F(\mathbf{x}) \leq nM$; therefore the number of steps is bounded by n^{2a-1} . \square

Lemma 3.8. *If \mathbf{x} is the output of Algorithm 3.6, then with high probability*

$$2F(\mathbf{x}) \geq F(\mathbf{x} \vee \mathbf{y}) + F(\mathbf{x} \wedge \mathbf{y}) - 5\delta n$$

for every $\mathbf{y} \in P$.

Proof. If the algorithm terminates, it means that for every $\mathbf{y} \in P$, $(\mathbf{y} - \mathbf{x}) \cdot \tilde{\nabla}F(\mathbf{x}) \leq 4\delta n$. Considering the accuracy of our estimate of the gradient $\tilde{\nabla}F(\mathbf{x})$ (with high probability), this means that $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \leq 5\delta n$. By Lemma 3.2, we have $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \geq F(\mathbf{x} \vee \mathbf{y}) + F(\mathbf{x} \wedge \mathbf{y}) - 2F(\mathbf{x})$. This proves the lemma. \square

3.3 Repeated local search: a 0.25-approximation

Next, we show that how to design a 0.25-approximation to the multilinear optimization problem using two runs of the fractional local-search algorithm. The following is our algorithm.

Algorithm 3.9. *Let \mathbf{x} be the output of Algorithm 3.6 on the polytope P . Define $Q = \{\mathbf{y} \in P : \mathbf{y} \leq \mathbf{1} - \mathbf{x}\}$ and let \mathbf{z} be the output of Algorithm 3.6 on the polytope Q . Return the better of $F(\mathbf{x})$ and $F(\mathbf{z})$.*

We use the following property of the multilinear extension of a submodular function. Let us replace each coordinate by a $[0, 1]$ interval and let us represent a certain value x_i of the i 'th coordinate by a subset of $[0, 1]$ of the corresponding measure.

Definition 3.10. Let $\mathcal{X} \in \mathcal{L}^N$, where \mathcal{L} denotes the set of all measurable subsets of $[0, 1]$. We say that \mathcal{X} represents a vector $\mathbf{x} \in [0, 1]^N$, if \mathcal{X}_i has measure x_i for each $i \in N$.

From a "discrete point of view", we can imagine that each coordinate is replaced by some large number of elements M and a value of x_i is represented by any subset of size Mx_i . This can be carried out if all the vectors we work with are rational. In the following, we consider functions on subsets of this new ground set. We show a natural property, namely that a function derived from the multilinear extension of a submodular function is again submodular. (An analogous property in the discrete case was proved in [40, 36].)

Lemma 3.11. Let $F : [0, 1]^N \rightarrow \mathbb{R}$ be a multilinear extension of a submodular function f . Define a function F^* on \mathcal{L}^N , by $F^*(\mathcal{X}) = F(\mathbf{x})$, where $\mathbf{x} \in [0, 1]^N$ is the vector represented by \mathcal{X} . Then F^* is submodular:

$$F^*(\mathcal{X} \cup \mathcal{Y}) + F^*(\mathcal{X} \cap \mathcal{Y}) \leq F^*(\mathcal{X}) + F^*(\mathcal{Y}),$$

where the union and intersection is interpreted component-wise.

Proof. We have $F(\mathbf{x}) = \mathbf{E}[f(\hat{\mathbf{x}})]$ where $\hat{x}_i = 1$ independently with probability x_i . An equivalent way to generate $\hat{\mathbf{x}}$ is to choose any set $\mathcal{X} \in \mathcal{L}^N$ representing \mathbf{x} , generate uniformly and independently a number $r_i \in [0, 1]$ for each $i \in N$, and set $\hat{x}_i = 1$ iff $r_i \in \mathcal{X}_i$. Since the measure of \mathcal{X}_i is x_i , $\hat{x}_i = 1$ with probability exactly x_i . Therefore,

$$F^*(\mathcal{X}) = F(\mathbf{x}) = \mathbf{E}[f(\hat{\mathbf{x}})] = \mathbf{E}[f(\{i : r_i \in \mathcal{X}_i\})].$$

Similarly,

$$F^*(\mathcal{Y}) = \mathbf{E}[f(\{i : r_i \in \mathcal{Y}_i\})].$$

This also holds for $\mathcal{X} \cup \mathcal{Y}$ and $\mathcal{X} \cap \mathcal{Y}$: since $(\mathcal{X} \cup \mathcal{Y})_i = \mathcal{X}_i \cup \mathcal{Y}_i$ and $(\mathcal{X} \cap \mathcal{Y})_i = \mathcal{X}_i \cap \mathcal{Y}_i$, we get

$$F^*(\mathcal{X} \cup \mathcal{Y}) = \mathbf{E}[f(\{i : r_i \in \mathcal{X}_i \cup \{i : r_i \in \mathcal{Y}_i\})]$$

and

$$F^*(\mathcal{X} \cap \mathcal{Y}) = \mathbf{E}[f(\{i : r_i \in \mathcal{X}_i \cap \{i : r_i \in \mathcal{Y}_i\})].$$

Hence, by the submodularity of f ,

$$\begin{aligned} F^*(\mathcal{X} \cup \mathcal{Y}) + F^*(\mathcal{X} \cap \mathcal{Y}) &= \mathbf{E}[f(\{i : r_i \in \mathcal{X}_i \cup \{i : r_i \in \mathcal{Y}_i\}) + f(\{i : r_i \in \mathcal{X}_i \cap \{i : r_i \in \mathcal{Y}_i\})] \\ &\leq \mathbf{E}[f(\{i : r_i \in \mathcal{X}_i\}) + f(\{i : r_i \in \mathcal{Y}_i\})] \\ &= F^*(\mathcal{X}) + F^*(\mathcal{Y}). \end{aligned}$$

□

From here, we obtain our main lemma - the average of the two fractional local optima is at least $\frac{1}{4}\text{OPT}$.

Lemma 3.12. Let $\text{OPT} = \max\{F(\mathbf{x}) : \mathbf{x} \in P\}$. Let \mathbf{x} be the output of Algorithm 3.6 on polytope P , and \mathbf{z} an output of Algorithm 3.6 on polytope $Q = \{\mathbf{y} \in P : \mathbf{y} \leq \mathbf{1} - \mathbf{x}\}$, with parameter δ as in Algorithm 3.6. Then with high probability, $2F(\mathbf{x}) + 2F(\mathbf{z}) \geq \text{OPT} - 10\delta n$.

Proof. Let $\text{OPT} = F(\mathbf{x}^*)$ where $\mathbf{x}^* \in P$. By Lemma 3.8, the output of the algorithm \mathbf{x} satisfies with high probability

$$2F(\mathbf{x}) \geq F(\mathbf{x} \vee \mathbf{x}^*) + F(\mathbf{x} \wedge \mathbf{x}^*) - 5\delta n. \quad (1)$$

In the restricted polytope $Q = \{\mathbf{y} \in P : \mathbf{y} \leq \mathbf{1} - \mathbf{x}\}$, consider the point $\mathbf{z}^* = (\mathbf{x}^* - \mathbf{x}) \vee \mathbf{0} \in Q$. Again by Lemma 3.8, the output of the algorithm \mathbf{z} satisfies

$$2F(\mathbf{z}) \geq F(\mathbf{z} \vee \mathbf{z}^*) + F(\mathbf{z} \wedge \mathbf{z}^*) - 5\delta n. \quad (2)$$

Now we use a representation of vectors by subsets as described in Def. 3.10. We choose $\mathcal{X}, \mathcal{X}^*, \mathcal{Z}, \mathcal{Z}^* \in \mathcal{L}^N$ to represent $\mathbf{x}, \mathbf{x}^*, \mathbf{z}, \mathbf{z}^*$ as follows: for each $i \in N$, $\mathcal{X}_i = [0, x_i]$, $\mathcal{Z}_i = [x_i, x_i + z_i]$ (note that $x_i + z_i \leq 1$), $\mathcal{X}_i^* = [0, x_i^*]$ and $\mathcal{Z}_i^* = [0, z_i^*] = [0, \max\{x_i^* - x_i, 0\}]$. Note that $(\mathcal{X} \cap \mathcal{Z})_i = \emptyset$ for all $i \in N$.

Defining F^* as in Lemma 3.11, we have $F^*(\mathcal{X}) = F(\mathbf{x})$, $F^*(\mathcal{X}^*) = F(\mathbf{x}^*) = \text{OPT}$, $F^*(\mathcal{Z}) = F(\mathbf{z})$ and $F^*(\mathcal{Z}^*) = F(\mathbf{z}^*)$. Using relations like $[0, x_i] \cup [0, x_i^*] = [0, \max\{x_i, x_i^*\}]$, we also get $F^*(\mathcal{X} \cup \mathcal{X}^*) = F(\mathbf{x} \vee \mathbf{x}^*)$ and $F^*(\mathcal{X} \cap \mathcal{X}^*) = F(\mathbf{x} \wedge \mathbf{x}^*)$. Furthermore, we have $(\mathcal{X}_i^* \setminus \mathcal{X}_i) \cup \mathcal{Z}_i = [x_i, \max\{x_i^*, x_i + z_i\}] = [x_i, x_i + \max\{z_i^*, z_i\}]$. This is an interval of length $\max\{z_i^*, z_i\} = (\mathbf{z} \vee \mathbf{z}^*)_i$ and hence $F^*((\mathcal{X}^* \setminus \mathcal{X}) \cup \mathcal{Z}) = F(\mathbf{z} \vee \mathbf{z}^*)$, where $(\mathcal{X}^* \setminus \mathcal{X}) \cup \mathcal{Z}$ is interpreted component-wise.

The property of the first local optimum (1) can be thus written as $2F(\mathbf{x}) \geq F^*(\mathcal{X} \cup \mathcal{X}^*) + F^*(\mathcal{X} \cap \mathcal{X}^*) - 5\delta n$. The property of the complementary local optimum (2) can be written as $2F(\mathbf{z}) \geq F^*((\mathcal{X}^* \setminus \mathcal{X}) \cup \mathcal{Z}) - 5\delta n$ (we discarded the nonnegative term $F(\mathbf{z} \wedge \mathbf{z}^*)$ which is not used in the following). Therefore, $2F(\mathbf{x}) + 2F(\mathbf{z}) \geq F^*(\mathcal{X} \cup \mathcal{X}^*) + F^*(\mathcal{X} \cap \mathcal{X}^*) + F^*((\mathcal{X}^* \setminus \mathcal{X}) \cup \mathcal{Z}) - 10\delta n$. By Lemma 3.11, F^* is submodular. Hence we get

$$\begin{aligned} F^*(\mathcal{X} \cap \mathcal{X}^*) + F^*((\mathcal{X}^* \setminus \mathcal{X}) \cup \mathcal{Z}) &\geq F^*((\mathcal{X} \cap \mathcal{X}^*) \cup ((\mathcal{X}^* \setminus \mathcal{X}) \cup \mathcal{Z})) \\ &= F^*(\mathcal{X}^* \cup \mathcal{Z}) \end{aligned}$$

(we discarded the intersection term). Finally, using the fact that $\mathcal{X} \cap \mathcal{Z} = \emptyset$ and again the submodularity of F^* , we get

$$F^*(\mathcal{X} \cup \mathcal{X}^*) + F^*(\mathcal{X}^* \cup \mathcal{Z}) \geq F^*((\mathcal{X} \cup \mathcal{X}^*) \cap (\mathcal{X}^* \cup \mathcal{Z})) = F^*(\mathcal{X}^*)$$

(we discarded the union term). To summarize,

$$\begin{aligned} 2F(\mathbf{x}) + 2F(\mathbf{z}) &\geq F^*(\mathcal{X} \cup \mathcal{X}^*) + F^*(\mathcal{X} \cap \mathcal{X}^*) + F^*((\mathcal{X}^* \setminus \mathcal{X}) \cup \mathcal{Z}) - 10\delta n \\ &\geq F^*(\mathcal{X} \cup \mathcal{X}^*) + F^*(\mathcal{X}^* \cup \mathcal{Z}) - 10\delta n \\ &\geq F^*(\mathcal{X}^*) - 10\delta n = \text{OPT} - 10\delta n. \end{aligned}$$

□

Since the parameter δ in Algorithm 3.6 is chosen as $\delta = \frac{M}{n^{a-1}}$ for some constant $a > 3$, we obtain the following.

Corollary 3.13. *For any solvable down-monotone polytope $P \subseteq [0, 1]^N$ and multilinear extension of a submodular function $F : [0, 1]^N \rightarrow \mathbb{R}_+$, Algorithm 3.9 finds with high probability a solution of value at least $\frac{1}{4}\text{OPT} - O(\frac{M}{n^{a-2}})$ for the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$.*

We remark that in many settings of interest, $\text{OPT} = \max\{F(\mathbf{x}) : \mathbf{x} \in P\} \geq M/\text{poly}(n)$ and thus we can make the additive error small relative to the optimum by choosing a large enough. This leads to a multiplicative $(1/4 - o(1))$ -approximation. A concrete setting of interest is when P is not too thin in any dimension, as highlighted by the following lemma which, together with Corollary 3.13, implies Theorem 1.1.

Lemma 3.14. *Let $f : 2^N \rightarrow \mathbb{R}_{\geq 0}$ be a nonnegative submodular function with multilinear extension F , and let $P \subseteq \mathbb{R}^n$ be a solvable down-monotone polytope satisfying that there is a $\lambda \in \Omega(\frac{1}{\text{poly}(n)})$ such that for each coordinate $i \in [n]$, $\lambda \cdot \mathbf{e}_i \in P$. Furthermore, let $\text{OPT} = \max\{F(x) \mid x \in P\}$, and $M = \max\{f(i), f(N-i) : i \in N\}$. Then*

$$\text{OPT} \geq \Omega\left(\frac{M}{\text{poly}(n)}\right).$$

Proof. Let $i \in N$. Since F is concave along any nonnegative direction and $\lambda \mathbf{e}_i \in P$, we have

$$\text{OPT} \geq F(\lambda \mathbf{e}_i) \geq \lambda F(\mathbf{e}_i) = \lambda f(i) \quad \forall i \in N. \quad (3)$$

Furthermore,

$$f(N-i) \leq \sum_{j \in N-i} f(j) \leq \frac{n}{\lambda} \text{OPT}, \quad (4)$$

where we used (3) for the last inequality. Equations (3) and (4) imply the desired results. □

3.4 Restricted local search: a 0.309-approximation

Next, we present a modified local-search algorithm which is a generalization of the algorithm for matroid polytopes from [50]. We remark that this algorithm is in fact simpler than the $\frac{1}{4}$ -approximation from the previous section, in that it does not require a second-stage complementary local search. Both algorithms work for any down-monotone polytope P . However, our analysis of the restricted local-search algorithm is with respect to the best integer solution in the polytope; we do not know whether the approximation guarantee holds with respect to the best fractional solution.

Algorithm 3.15. Fix a parameter $t \in [0, 1]$. Using Algorithm 3.6, find an approximate local optimum \mathbf{x} in the polytope $P \cap [0, t]^N$. Return \mathbf{x} .

We show that with the choice of $t = \frac{1}{2}(3 - \sqrt{5})$, this algorithm achieves a $\frac{1}{4}(-1 + \sqrt{5}) \simeq 0.309$ -approximation with respect to the optimal integer solution in P .

Lemma 3.16. Let \mathbf{x} be an output of Algorithm 3.6 on $P \cap [0, t]^N$. Define $\mathbf{w} \in [0, 1]^N$ by $w_i = t$ if $x_i \geq t - 1/n$ and $w_i = 1$ if $x_i < t - 1/n$. Let \mathbf{z} be any point in P and let $\mathbf{z}' = \mathbf{z} \wedge \mathbf{w}$. Then with high probability,

$$2F(\mathbf{x}) \geq F(\mathbf{x} \vee \mathbf{z}') + F(\mathbf{x} \wedge \mathbf{z}') - 5\delta n^2.$$

We remark that the above inequality would be immediate from Lemma 3.8, if $\mathbf{z}' \in P \cap [0, t]^N$. However, \mathbf{z}' is not necessarily constrained by $[0, t]^N$.

Proof. Consider $\mathbf{z}' = \mathbf{z} \wedge \mathbf{w}$ as defined above. By down-monotonicity, $\mathbf{z}' \in P$. Also, the coordinates where $z'_i > t$ are exactly those where $x_i < t - 1/n$. So we have $\mathbf{x} + \frac{1}{n}(\mathbf{z}' - \mathbf{x}) \in P \cap [0, t]^N$. By the stopping rule of Algorithm 3.6,

$$\frac{1}{n}(\mathbf{z}' - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \leq 5\delta n.$$

By Lemma 3.2, this implies $F(\mathbf{x} \vee \mathbf{z}') + F(\mathbf{x} \wedge \mathbf{z}') - 2F(\mathbf{x}) \leq (\mathbf{z}' - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \leq 5\delta n^2$. \square

In the rest of the analysis, we follow [50].

Definition 3.17. For $\mathbf{x} \in [0, 1]^N$ and $\lambda \in [0, 1]$, we define the associated “threshold set” as $T_{>\lambda}(\mathbf{x}) = \{i : x_i > \lambda\}$.

Lemma 3.18. Let $\mathbf{x} \in [0, 1]^N$. For any partition $N = C \cup \bar{C}$,

$$F(\mathbf{x}) \geq \mathbf{E}[f((T_{>\lambda}(\mathbf{x}) \cap C) \cup (T_{>\lambda'}(\mathbf{x}) \cap \bar{C}))]$$

where $\lambda, \lambda' \in [0, 1]$ are independently and uniformly random.

This appears as Lemma A.5 in [50]. We remark that the right-hand side with $C = \emptyset$ or $C = N$ gives the Lovász extension of f and the lemma follows by comparing the multilinear and Lovász extension. For a non-trivial partition (C, \bar{C}) , the lemma follows by two applications of this fact. The next lemma is exactly as in [50] for the special case of a matroid polytope; we rephrase the proof here in our more general setting.

Lemma 3.19. Assume that $t \in [0, \frac{1}{2}(3 - \sqrt{5})]$. Let \mathbf{x} be an output of Algorithm 3.6 on $P \cap [0, t]^N$ (with parameter $a \geq 4$), and let $\mathbf{z} = \mathbf{1}_C$ be any integer solution in P . Then with high probability,

$$F(\mathbf{x}) \geq \left(t - \frac{1}{2}t^2\right) f(C) - O\left(\frac{M}{n^{a-3}}\right).$$

Proof. Define $A = \{i : x_i \geq t - 1/n\}$ and let $\mathbf{w} = t\mathbf{1}_A + \mathbf{1}_{\bar{A}}$, $\mathbf{z}' = \mathbf{z} \wedge \mathbf{w}$ as in Lemma 3.16. Since $\mathbf{z} = \mathbf{1}_C$, we have $\mathbf{z}' = t\mathbf{1}_{A \cap C} + \mathbf{1}_{C \setminus A}$. By Lemma 3.16, we get

$$2F(\mathbf{x}) \geq F(\mathbf{x} \vee \mathbf{z}') + F(\mathbf{x} \wedge \mathbf{z}') - 5\delta n^2. \tag{5}$$

First, let us analyze $F(\mathbf{x} \wedge \mathbf{z}')$. Since $\mathbf{z}' = t\mathbf{1}_{A \cap C} + \mathbf{1}_{C \setminus A}$ and $\mathbf{x} \in [0, t]^N$, we have $\mathbf{x} \wedge \mathbf{z}' = \mathbf{x} \wedge \mathbf{1}_C$. We apply Lemma 3.18, which states that

$$F(\mathbf{x} \wedge \mathbf{z}') = F(\mathbf{x} \wedge \mathbf{1}_C) \geq \mathbf{E}[f(T_{>\lambda}(\mathbf{x}) \cap C)].$$

Due to the definition of $T_{>\lambda}(\mathbf{x})$, with probability $t - 1/n$ we have $\lambda < t - 1/n$ and $T_{>\lambda}(\mathbf{x})$ contains $A = \{i : x_i \geq t - 1/n\}$. Then, $f(T_{>\lambda}(\mathbf{x}) \cap C) + f(C \setminus A) \geq f(C)$ by submodularity. We conclude that

$$F(\mathbf{x} \wedge \mathbf{z}') \geq \left(t - \frac{1}{n}\right) (f(C) - f(C \setminus A)). \quad (6)$$

Next, let us analyze $F(\mathbf{x} \vee \mathbf{z}')$. We apply Lemma 3.18. We get

$$F(\mathbf{x} \vee \mathbf{z}') \geq \mathbf{E}[f((T_{>\lambda}(\mathbf{x} \vee \mathbf{z}') \cap C) \cup (T_{>\lambda'}(\mathbf{x} \vee \mathbf{z}') \cap \bar{C}))].$$

The random threshold sets are as follows: $T_{>\lambda}(\mathbf{x} \vee \mathbf{z}') \cap C = T_{>\lambda}(\mathbf{z}')$ is equal to C with probability t , and equal to $C \setminus A$ with probability $1 - t$, by the definition of \mathbf{z}' . $T_{>\lambda'}(\mathbf{x} \vee \mathbf{z}') \cap \bar{C} = T_{>\lambda'}(\mathbf{x}) \cap \bar{C}$ is empty with probability $1 - t$, because $\mathbf{x} \in [0, t]^N$. (We ignore the contribution when $T_{>\lambda'}(\mathbf{x}) \cap \bar{C} \neq \emptyset$.) Because λ, λ' are independently sampled, we get

$$F(\mathbf{x} \vee \mathbf{z}') \geq (1 - t)(tf(C) + (1 - t)f(C \setminus A)).$$

Provided that $t \in [0, \frac{1}{2}(3 - \sqrt{5})]$, we have $t \leq (1 - t)^2$. Then, we can write

$$F(\mathbf{x} \vee \mathbf{z}') \geq t(1 - t)f(C) + tf(C \setminus A). \quad (7)$$

Combining equations (5), (6) and (7), we get

$$\begin{aligned} 2F(\mathbf{x}) &\geq F(\mathbf{x} \vee \mathbf{z}') + F(\mathbf{x} \wedge \mathbf{z}') - 5\delta n^2 \\ &\geq t(1 - t)f(C) + tf(C \setminus A) + (t - \frac{1}{n})(f(C) - f(C \setminus A)) - 5\delta n^2 \\ &\geq (2t - t^2)f(C) - O\left(\frac{M}{n^{a-3}}\right) \end{aligned}$$

using $\delta n^2 = M/n^{a-3}$. □

Next, we show how the error term in Lemma 3.19 can be compared to the optimal value. Note that here we use the fact that in this section, we compare to 0/1 solutions only. The following Lemma is essentially a specialization of Lemma 3.14 to the 0/1 case.

Lemma 3.20. *Suppose that $\mathbf{e}_i \in P$ for each $i \in N$. Let $OPT = \max\{F(\mathbf{x}) : \mathbf{x} \in P \cap \{0, 1\}^N\}$ and $M = \max_{i \in N}\{f(i), f(N - i)\}$. Then $OPT \geq \frac{1}{n}M$.*

Proof. If $M = f(i)$ for some $i \in N$, then clearly $OPT = \max\{F(\mathbf{x}) : \mathbf{x} \in P \cap \{0, 1\}^N\} \geq M$, because $F(\mathbf{e}_i) = f(i)$ and $\mathbf{e}_i \in P$. If $M = f(N - i)$ for some $i \in N$, then consider $\sum_{j \neq i} f(j) \geq f(N - i) = M$ which holds by submodularity and nonnegativity of f . We have $f(j) \geq \frac{1}{n}M$ for some $j \neq i$. By the above argument, $OPT \geq \frac{1}{n}M$. □

Clearly, if $\mathbf{e}_i \notin P$, then coordinate x_i cannot participate in an integer optimum, $\max\{F(\mathbf{x}) : \mathbf{x} \in P \cap \{0, 1\}^N\}$. We can remove all such coordinates from the problem. Therefore, we can in fact assume that $\mathbf{e}_i \in P$ for all $i \in N$.

Corollary 3.21. *Assume $\mathbf{e}_i \in P$ for all $i \in N$. Then for $t = \frac{1}{2}(3 - \sqrt{5})$, Algorithm 3.15 with high probability a $\frac{1}{4}(-1 + \sqrt{5} - o(1))$ -approximation for the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P \cap \{0, 1\}^N\}$.*

4 Contention resolution schemes

In this section we discuss contention resolution schemes in more detail and prove our results on the existence of contention resolution schemes and their application to submodular maximization problems.

4.1 Contention resolution basics

Recall the definition, from Section 1, of a (b, c) -balanced CR scheme π for a polytope $P_{\mathcal{I}}$. We first prove the claim that a (b, c) -balanced CR scheme π can be transformed into a bc -balanced CR scheme π' as follows. Let $\mathbf{x} \in P_{\mathcal{I}}$ and $A \subseteq N$. We define $\pi'_{\mathbf{x}}(A)$ as follows. First each element of A is removed independently of the others with probability $1 - b$ to obtain a random set $A' \subseteq A$. We then set $\pi'_{\mathbf{x}}(A) = \pi_{\mathbf{x}}(A')$. The key observation is that if A is a set drawn according to the distribution induced by $R(\mathbf{x})$, then A' has a distribution given by $R(b\mathbf{x})$. Hence, for any $i \in N$

$$\begin{aligned} \Pr[i \in \pi'_{\mathbf{x}}(R(\mathbf{x})) \mid i \in R(\mathbf{x})] &= \frac{\Pr[i \in \pi'_{\mathbf{x}}(R(\mathbf{x}))]}{\Pr[i \in R(\mathbf{x})]} = \frac{\Pr[i \in \pi_{\mathbf{x}}(R(b\mathbf{x}))]}{\Pr[i \in R(\mathbf{x})]} = \frac{b \Pr[i \in \pi_{\mathbf{x}}(R(b\mathbf{x}))]}{\Pr[i \in R(b\mathbf{x})]} \\ &= b \Pr[i \in \pi_{\mathbf{x}}(R(b\mathbf{x})) \mid i \in R(b\mathbf{x})] \geq bc, \end{aligned}$$

where the last inequality follows from the fact that π is (b, c) -balanced.

Monotonicity of CR schemes for submodular function maximization: The inequality that relates contention resolution to submodular maximization is given in Theorem 1.3. A proof of this inequality also appears in [4] for monotone functions without the pruning procedure. Before presenting the proof, we provide some intuition on why monotonicity of the CR scheme is needed in the context of submodular function maximization, and we specify the pruning procedure η_f . It is easy to see that if $P_{\mathcal{I}}$ has a c -balanced CR scheme then it implies a c -approximation for maximizing a linear function over $P_{\mathcal{I}}$. If \mathbf{x} is a fractional solution then its value is $\sum_i w_i x_i$, where w_i are some (non-negative) weights; since each element i is present in the final solution produced by a c -balanced CR scheme with probability at least $c x_i$, by linearity of expectation, the expected weight of a solution returned by a c -balanced scheme is at least $c \sum_i w_i x_i$. More generally, we would like to prove such a bound for any submodular function f via F . However, this is no longer obvious since elements do not appear independently in the rounding scheme; recall that $F(\mathbf{x})$ is the expected value of f on a set produced by independently including each i with probability x_i . Monotonicity is the property that is useful in this context, because elements of smaller sets contribute more to a submodular function than elements of larger sets.

To prove Theorem 1.3, we first introduce the claimed pruning function η_f . To prune a set I via the pruning function η_f , an arbitrary ordering of the elements of N is fixed: for notational simplicity let $N = \{1, \dots, n\}$ which gives a natural ordering. Starting with $J = \emptyset$ the final set $J = \eta_f(I)$ —which we called the *pruned* version of I —is constructed by going through all elements of I in the order induced by N . When considering an element i , J is replaced by $J + i$ if $f(J + i) - f(J) > 0$.

Proof of Theorem 1.3. Let $R = R(\mathbf{x})$ and $I = \pi_{\mathbf{x}}(R)$, and let $J = \eta_f(I)$ if f is non-monotone and $J = I$ otherwise. Hence, in both cases, J is the set returned by the suggested rounding procedure.

Assume that $N = \{1, \dots, n\}$ is the same ordering of the elements as used in the pruning operation (in case no pruning was applied, any order is fine). The main property we get by pruning is the following. Notice that this property trivially holds when f is monotone.

$$f_{J \cap [i-1]}(i) \geq 0 \quad \forall i \in J. \quad (8)$$

Furthermore, for each $i \in I$

$$f_{J \cap [i-1]}(i) > 0 \quad \Rightarrow \quad i \in J. \quad (9)$$

Again, notice that this property holds trivially in the monotone case in which we have $J = I$.

The main step that we will prove is that for any fixed $i \in \{1, \dots, n\}$,

$$\mathbf{E}[f(J \cap [i]) - f(J \cap [i-1])] \geq c \mathbf{E}[f(R \cap [i]) - f(R \cap [i-1])]. \quad (10)$$

We highlight that there are two sources of randomness over which the expectation is taken on the left-hand side of the above inequality: one source is the randomness in choosing the set R , and the other source is the potential randomness of the CR scheme used to obtain the set I from R , which is later deterministically pruned to get J . The theorem then follows from (10) since

$$\mathbf{E}[f(J)] = f(\emptyset) + \sum_{i=1}^n \mathbf{E}[f(J \cap [i]) - f(J \cap [i-1])] \geq f(\emptyset) + c \sum_{i=1}^n \mathbf{E}[f(R \cap [i]) - f(R \cap [i-1])] \geq c \mathbf{E}[f(R)].$$

Hence, it remains to prove (10). Consider first the non-monotone case. Here we have

$$\begin{aligned}
\mathbf{E}[f(J \cap [i]) - f(J \cap [i-1])] &= \mathbf{E}[\mathbf{1}_{i \in J} f_{J \cap [i-1]}(i)] \\
&= \Pr[i \in R] \cdot \mathbf{E}[\mathbf{1}_{i \in J} f_{J \cap [i-1]}(i) \mid i \in R] \\
&\stackrel{(8)}{=} \Pr[i \in R] \cdot \mathbf{E}[\mathbf{1}_{i \in J} \max\{0, f_{J \cap [i-1]}(i)\} \mid i \in R] \\
&\stackrel{(9)}{=} \Pr[i \in R] \cdot \mathbf{E}[\mathbf{1}_{i \in I} \max\{0, f_{J \cap [i-1]}(i)\} \mid i \in R] \\
&\geq \Pr[i \in R] \cdot \mathbf{E}[\mathbf{1}_{i \in I} \max\{0, f_{R \cap [i-1]}(i)\} \mid i \in R] \quad (\text{since } f \text{ is submodular}) \\
&= \Pr[i \in R] \cdot \mathbf{E}[\mathbf{E}[\mathbf{1}_{i \in I} \max\{0, f_{R \cap [i-1]}(i)\} \mid R] \mid i \in R] \\
&= \Pr[i \in R] \cdot \mathbf{E}[\mathbf{E}[\mathbf{1}_{i \in I} \mid R] \max\{0, f_{R \cap [i-1]}(i)\} \mid i \in R].
\end{aligned}$$

On the product space associated with the distribution of R conditioned on $i \in R$, both of the terms $\mathbf{E}[\mathbf{1}_{i \in I} \mid R]$ and $\max\{0, f_{R \cap [i-1]}(i)\}$ are non-increasing functions, because of the monotonicity of the CR scheme used to obtain I from R and f being submodular, respectively. Notice that the randomness in both terms $\mathbf{E}[\mathbf{1}_{i \in I} \mid R]$ and $\max\{0, f_{R \cap [i-1]}(i)\}$ stems only from the random set R , and not from the potential randomness of the CR scheme. Hence, by the FKG inequality we obtain

$$\begin{aligned}
\Pr[i \in R] \cdot \mathbf{E}[\mathbf{E}[\mathbf{1}_{i \in I} \mid R] \max\{0, f_{R \cap [i-1]}(i)\} \mid i \in R] \\
&\geq \Pr[i \in R] \cdot \mathbf{E}[\mathbf{1}_{i \in I} \mid i \in R] \cdot \mathbf{E}[\max\{0, f_{R \cap [i-1]}(i)\} \mid i \in R] \\
&= \Pr[i \in R] \cdot \Pr[i \in I \mid i \in R] \cdot \mathbf{E}[\max\{0, f_{R \cap [i-1]}(i)\} \mid i \in R] \\
&\geq c \Pr[i \in R] \cdot \mathbf{E}[\max\{0, f_{R \cap [i-1]}(i)\} \mid i \in R] \\
&\geq c \Pr[i \in R] \cdot \mathbf{E}[f_{R \cap [i-1]}(i) \mid i \in R] \\
&= c \Pr[i \in R] \cdot \mathbf{E}[f_{R \cap [i-1]}(i)] \\
&= c \mathbf{E}[f(R \cap [i]) - f(R \cap [i-1])],
\end{aligned}$$

where in the second to last equality we use again the property that $f_{R \cap [i-1]}(i)$ is independent of $i \in R$. Hence, this shows (10) as desired, and completes the proof. \square

The following subsection on strict CR schemes discusses an alternate way of rounding that does not rely on pruning and is oblivious to the underlying submodular function. As we highlight below, such a procedure is useful when the value of several submodular functions should approximately be preserved, simultaneously. However, since we do not rely on this alternate procedure later, this part can safely be skipped.

Strict contention resolution schemes An alternative way to round in the context of non-monotone submodular functions, that does not rely on pruning, can be obtained by using a stronger notion of CR schemes. More precisely, we say that a (b, c) -balanced CR scheme π for $P_{\mathcal{I}}$ is *strict*, if it satisfies the second condition of a CR scheme with equality, i.e., $\Pr[i \in \pi_{\mathbf{x}}(R(\mathbf{x}))] = c$. We have the following (the proof can be found in Appendix B).

Theorem 4.1. *Let $f : 2^N \rightarrow \mathbb{R}_+$ be a non-negative submodular function with multilinear relaxation F , and \mathbf{x} be a point in $P_{\mathcal{I}}$, a convex relaxation for $\mathcal{I} \subseteq 2^N$. Let π be a monotone and strict (b, c) -balanced CR scheme for $P_{\mathcal{I}}$, and let $I = \pi_{\mathbf{x}}(R(\mathbf{x}))$. Then*

$$\mathbf{E}[f(I)] \geq c F(\mathbf{x}).$$

The advantage of using a strict CR scheme compared to applying the pruning step is that this version of rounding is oblivious to the underlying submodular function f . This could potentially be useful in settings where one is interested in simultaneously maximizing more than one submodular function. Assume for example that \mathbf{x} is a point such that $F_1(\mathbf{x})$ and $F_2(\mathbf{x})$ have simultaneously high values, where F_1 and F_2 are the multilinear relaxations of two submodular functions f_1 and f_2 . Then using a rounding that is oblivious to the underlying submodular function leads to a randomly rounded set I satisfying $\mathbf{E}[f_1(I)] \geq c F_1(\mathbf{x})$ and $\mathbf{E}[f_2(I)] \geq c F_2(\mathbf{x})$.

Any monotone but not necessarily strict (b, c) -balanced CR scheme π can be transformed into a monotone (b, c) -balanced CR scheme that is arbitrarily close to being strict as follows. For each element $i \in N$, one can estimate the probability $c'_i = \Pr[i \in \pi_{\mathbf{x}}(R(\mathbf{x})) \mid i \in I] \geq c$ via Monte-Carlo sampling within a polynomially small error

(assuming that c is a constant). Then we can modify the CR scheme by removing from its output I , element $i \in I$ with probability $1 - c/c'_i$. The resulting scheme is arbitrarily close to being strict and can be used in place of a strict scheme in Theorem 4.1 with a weaker guarantee; in applications to approximation, the ratio is affected in the lower-order terms. We omit further details.

Combining CR schemes: Next, we discuss how to combine contention resolution schemes for different constraints. We consider a constraint $\mathcal{I} = \bigcap_{i=1}^h \mathcal{I}_i$ and its polyhedral relaxation $P_{\mathcal{I}} = \bigcap_i P_{\mathcal{I}_i}$, such that $P_{\mathcal{I}_i}$ has a monotone (b, c_i) -balanced CR scheme π^i . We produce a contention resolution scheme π for \mathcal{I} which works with respect to the natural combination of constraint relaxations — an intersection of the respective polytopes $P_{\mathcal{I}_i}$. This ensures that the relaxed problem is still tractable and we can apply our optimization framework.

In case some elements $D \subseteq N$ are not part of the constraint \mathcal{I}_i , we assume without loss of generality that π^i never removes elements in D , i.e., $\pi_{\mathbf{x}}^i(A) \cap D = A \cap D$ for any $\mathbf{x} \in bP_{\mathcal{I}_i}$ and $A \subseteq N$. The combined contention resolution scheme π for $P_{\mathcal{I}}$ is defined by

$$\pi_{\mathbf{x}}(A) = \bigcap_i \pi_{\mathbf{x}}^i(A) \quad \text{for } A \subseteq N, \mathbf{x} \in bP_{\mathcal{I}}.$$

A straightforward union bound would state that the combined scheme π is $(b, 1 - \sum_i (1 - c_i))$ -balanced for $P_{\mathcal{I}}$. Using the FKG inequality, we obtain a stronger result in this setting, namely a $(b, \prod_i c_i)$ -balanced scheme. Moreover, if each constraint admits a (b, c) -balanced scheme and each element participates in at most k constraints, then we obtain a (b, c^k) -balanced scheme. This is the statement of Lemma 1.6 which we prove here using the combined scheme π defined above.

Proof of Lemma 1.6. Let us consider the $\leq k$ constraints that element i participates in. For simplicity we assume $k = 2$; the general statement follows by induction. For notational convenience we define $R = R(\mathbf{x})$, $I_1 = \pi_{\mathbf{x}}^1(R)$ and $I_2 = \pi_{\mathbf{x}}^2(R)$.

Conditioned on R , the choices of I_1, I_2 are independent, which means that

$$\Pr[i \in I_1 \cap I_2 \mid R] = \Pr[i \in I_1 \ \& \ i \in I_2 \mid R] = \Pr[i \in I_1 \mid R] \Pr[i \in I_2 \mid R].$$

Taking an expectation over R conditioned on $i \in R$, we get

$$\Pr[i \in I_1 \cap I_2 \mid i \in R] = \mathbf{E}_R[\Pr[i \in I_1 \cap I_2 \mid R] \mid i \in R] = \mathbf{E}_R[\Pr[i \in I_1 \mid R] \Pr[i \in I_2 \mid R] \mid i \in R].$$

Both $\Pr[i \in I_1 \mid R]$ and $\Pr[i \in I_2 \mid R]$ are non-increasing functions of R on the product space of sets containing i , so by the FKG inequality,

$$\mathbf{E}_R[\Pr[i \in I_1 \mid R] \Pr[i \in I_2 \mid R] \mid i \in R] \geq \mathbf{E}_R[\Pr[i \in I_1 \mid R] \mid i \in R] \cdot \mathbf{E}_R[\Pr[i \in I_2 \mid R] \mid i \in R].$$

Since these expectations are simply probabilities conditioned on $i \in R$, we conclude:

$$\Pr[i \in I_1 \cap I_2 \mid i \in R] \geq \Pr[i \in I_1 \mid i \in R] \Pr[i \in I_2 \mid i \in R].$$

Monotonicity of the above scheme is also easily implied: consider $j \in T_1 \subset T_2 \subseteq N$, then

$$\Pr[j \in I \mid R = T_1] = \prod_i \Pr[j \in I_i \mid R = T_1] \geq \prod_i \Pr[j \in I_i \mid R = T_2] = \Pr[j \in I \mid R = T_2].$$

where the inequality follows from the fact that each of the schemes is monotone. The polynomial time implementability of the composed scheme follows easily from the polynomial time implementability of π^1 and π^2 . \square

4.2 Obtaining CR schemes via distributions of deterministic CR schemes

We now describe a general way to obtain CR schemes relying on an LP approach. More precisely, we will observe that any CR scheme can be interpreted as a distribution over deterministic CR schemes. Exploiting this observation, we formulate an exponential-sized LP whose optimal solution corresponds to an optimal CR scheme. The separation problem of its dual then gives a natural characterization for the existence of strong CR schemes, which can be made

algorithmic in some interesting cases including matroid constraints, as we show in Section 4.4. Furthermore, in Section 4.3, we will use this point of view to draw a connection to a recently introduced concept, known as correlation gap.

Recall the formal definition of CR schemes given in Definition 1.2, in particular the differences between oblivious, deterministic and general (randomized) schemes. First, we note that the simplest CR schemes are the oblivious ones. An oblivious scheme does not depend on \mathbf{x} and is deterministic; hence it is essentially a single mapping $\pi : 2^N \rightarrow \mathcal{I}$ that given $A \subseteq N$ returns a set $\pi(A)$ such that $\pi(A) \subseteq A$ and $\pi(A) \in \mathcal{I}$. Several alteration based schemes are oblivious — see [5, 14] for some examples. A typical oblivious scheme fixes an ordering of the elements of N (that depends on the combinatorial properties of \mathcal{I}); it starts with an empty set A' , and considers the elements of A according to the fixed order and adds the current element i to the set A' if $A' \cup \{i\} \in \mathcal{I}$, otherwise it discards i . Finally it outputs A' . These greedy ordering based insertion schemes are easily seen to be monotone. A deterministic CR scheme is more general than an oblivious scheme in that the output can depend on \mathbf{x} ; in other words, for each $\mathbf{x} \in P_{\mathcal{I}}$, $\pi_{\mathbf{x}}$ is a mapping from 2^N to \mathcal{I} . The advantage or need for such a dependence is demonstrated by matroid polytopes. Let $P(\mathcal{M})$ be the convex hull of the independent sets of a matroid \mathcal{M} ; oblivious schemes cannot give a c -balanced CR scheme for any constant c . However, we can show that for any $b \in [0, 1]$ a good deterministic CR scheme exists: for any $\mathbf{x} \in P_{\mathcal{M}}$, there is an ordering $\sigma_{\mathbf{x}}$ that can be efficiently computed from \mathbf{x} such that a greedy insertion scheme based on the ordering $\sigma_{\mathbf{x}}$ gives a $(b, 1 - b)$ -balanced scheme. Such a scheme for $b = 1/2$ is implicitly present in [11], however for completeness, we give the details of our scheme in Section 4.4. The algorithm in [10] for geometric packing problems was reinterpreted as a deterministic CR scheme following our work; it is also based on computing an ordering that depends on \mathbf{x} followed by a greedy insertion procedure via the computed ordering (see also more recent work [19]). Such ordering based deterministic schemes are easily seen to be monotone.

In contrast to deterministic schemes, general (randomized) CR schemes are such that $\pi_{\mathbf{x}}(A)$ is a *random* feasible subset of A . Randomization is necessary to obtain an optimal result even when considering contention for a single item [22, 23]. For the time being, we do not require the CR schemes to be monotone; this is a point we discuss later. A non-oblivious (b, c) -balanced CR scheme π , deterministic or randomized, can depend on \mathbf{x} , and hence it is convenient to view it as a collection of separate schemes, one for each $\mathbf{x} \in bP_{\mathcal{I}}$. They are only tied together by the uniform guarantee c . In the following we will fix a particular \mathbf{x} and focus on finding the best scheme $\pi_{\mathbf{x}}$ for it. As we already discussed, if π is deterministic, then $\pi_{\mathbf{x}}$ is a mapping from 2^N to \mathcal{I} . We observe that a randomized scheme $\pi_{\mathbf{x}}$ is a distribution over deterministic schemes; note that here we are ignoring computational issues as well as monotonicity. We formalize this now. Call a mapping ϕ from 2^N to \mathcal{I} *valid* if $\phi(A) \subseteq A \forall A \subseteq N$. Let Φ^* be the family of all valid mappings from 2^N to \mathcal{I} . Any probability distribution (λ_{ϕ}) , $\phi \in \Phi^*$ induces a randomized scheme $\pi_{\mathbf{x}}$ as follows. For a set A , the algorithm $\pi_{\mathbf{x}}$ first picks $\phi \in \Phi^*$ according to the given probability distribution and then outputs $\phi(A)$. Conversely, for every randomized scheme $\pi_{\mathbf{x}}$, there is an associated probability distribution (λ_{ϕ}) , $\phi \in \Phi^*$ ¹⁰. Based on the preceding observation, one can write an LP to express the problem of finding a CR scheme that is (b, c) -balanced for \mathbf{x} with a value of c as high as possible. More precisely, for each $\phi \in \Phi^*$, we define $q_{i,\phi} = \Pr[i \in \phi(R)]$, where, as usual, $R := R(\mathbf{x})$ is obtained by including each $j \in N$ in R with probability x_j , independently of the other elements. Thus, for a given distribution $(\lambda_{\phi})_{\phi \in \Phi^*}$, the probability that the corresponding CR scheme $\pi_{\mathbf{x}}$ returns a set $\pi_{\mathbf{x}}(R)$ containing i , is given by $\sum_{\phi \in \Phi^*} q_{i,\phi} \lambda_{\phi}$. Hence, the problem of finding the distribution $(\lambda_{\phi})_{\phi \in \Phi^*}$ that leads to a (b, c) -balanced CR scheme for \mathbf{x} with c as high as possible can be formulated as the following linear program (LP1), with corresponding dual (DP1).

$$\begin{aligned}
\text{(LP1)} \quad & \max \quad c \\
& \text{s.t.} \quad \sum_{\phi \in \Phi^*} q_{i,\phi} \lambda_{\phi} \geq x_i c \quad \forall i \in N \\
& \quad \quad \sum_{\phi \in \Phi^*} \lambda_{\phi} = 1 \\
& \quad \quad \lambda_{\phi} \geq 0 \quad \forall \phi \in \Phi^* \\
\text{(DP1)} \quad & \min \quad \mu \\
& \text{s.t.} \quad \sum_{i \in N} q_{i,\phi} y_i \leq \mu \quad \forall \phi \in \Phi^* \\
& \quad \quad \sum_{i \in N} x_i y_i = 1 \\
& \quad \quad y_i \geq 0 \quad \forall i \in N
\end{aligned}$$

¹⁰Let k be an upper bound on the number of random bits used by $\pi_{\mathbf{x}}$. For any fixed string r of k random bits, let ϕ^r be the valid mapping from 2^N to \mathcal{I} generated by the algorithm $\pi_{\mathbf{x}}$ with random bits set to r . The distribution where for each r the probability assigned to ϕ^r is $1/2^k$ is the desired one.

In general we may also be interested in a restricted set of mappings $\Phi \subseteq \Phi^*$. In the above LP we can replace Φ^* by Φ to obtain the best c that can be achieved by taking probability distributions over valid mappings in Φ . Let $c(\mathbf{x}, \Phi)$ be the optimum value of the LP for a given \mathbf{x} and a set $\Phi \subseteq \Phi^*$. It is easy to see that $c(\mathbf{x}, \Phi) \leq c(\mathbf{x}, \Phi^*)$ for any Φ . From the earlier discussion, $c(\mathbf{x}, \Phi^*)$ is the best scheme for \mathbf{x} . We summarize the discussion so far by the following.

Proposition 4.2. *There exists a (b, c) -balanced CR scheme for $P_{\mathcal{I}}$ iff $\inf_{\mathbf{x} \in bP_{\mathcal{I}}} c(\mathbf{x}, \Phi^*) \geq c$.*

Proving the existence of a (b, c) -balanced CR scheme: To show that $P_{\mathcal{I}}$ has a (b, c) -balanced CR scheme we need to show that $c(\mathbf{x}, \Phi^*) \geq c$ for all $\mathbf{x} \in bP_{\mathcal{I}}$. By LP duality this is equivalent to showing that the optimum value of the dual (DP1) is at least c for all \mathbf{x} . We first reformulate the dual in a convenient form so that proving a lower bound c on the dual optimum reduces to a more intuitive question. We will then address the issue of efficiently constructing a CR scheme that nearly matches the lower bound.

Below we will use R to denote a random set obtained by picking each $i \in N$ independently with probability x_i and use probabilities and expectations with respect to this random process. The optimum value of the dual can be rewritten as:

$$\min_{\mathbf{y} \geq 0} \max_{\phi \in \Phi^*} \frac{\sum_i q_{i, \phi} y_i}{\sum_{i \in N} x_i y_i} = \min_{\mathbf{y} \geq 0} \max_{\phi \in \Phi^*} \frac{\sum_i y_i \Pr[i \in \phi(R)]}{\sum_{i \in N} x_i y_i} = \min_{\mathbf{y} \geq 0} \max_{\phi \in \Phi^*} \frac{\mathbf{E}_R \left[\sum_{i \in \phi(R)} y_i \right]}{\sum_{i \in N} x_i y_i}$$

For any fixed weight vector $\mathbf{y} \geq 0$ we claim that

$$\max_{\phi \in \Phi^*} \mathbf{E}_R \left[\sum_{i \in \phi(R)} y_i \right] = \mathbf{E}_R \left[\max_{S \subseteq R, S \in \mathcal{I}} \sum_{i \in S} y_i \right],$$

which follows by considering the specific mapping $\phi \in \Phi^*$ that for each $A \subseteq N$ sets $\phi(A) = \max_{A' \subseteq A, A' \in \mathcal{I}} y(A')$. Thus, the dual optimum value is

$$\min_{\mathbf{y} \geq 0} \frac{\mathbf{E}_R \left[\max_{S \subseteq R, S \in \mathcal{I}} \sum_{i \in S} y_i \right]}{\sum_{i \in N} x_i y_i}. \quad (11)$$

The above expression can be explained as an “integrality gap” of $P_{\mathcal{I}}$ for a specific rounding strategy; here the problem of interest is to find a maximum weight independent set in \mathcal{I} . The vector \mathbf{y} corresponds to weights on N . The vector \mathbf{x} corresponds to a fractional solution in $bP_{\mathcal{I}}$ (it is helpful here to think of $b = 1$). Thus $\sum_{i \in N} x_i y_i$ is the value of the fractional solution. The numerator is the expected value of a maximum weight independent set in R . Since we are minimizing over \mathbf{y} , the ratio is the worst case gap between the value of an integral feasible solution (obtained via a specific rounding) and a fractional solution.

Thus, to prove the existence of a (b, c) -balanced CR scheme it is sufficient (and necessary) to prove that for all $\mathbf{y} \geq 0$ and $\mathbf{x} \in bP_{\mathcal{I}}$

$$\frac{\mathbf{E}_{R(\mathbf{x})} \left[\max_{S \subseteq R, S \in \mathcal{I}} \sum_{i \in S} y_i \right]}{\sum_{i \in N} x_i y_i} \geq c.$$

Constructing CR schemes via the ellipsoid algorithm: We now discuss how to efficiently compute the best CR scheme for a given \mathbf{x} by solving (LP1) via the dual (DP1). We observe that $c(\mathbf{x}, \Phi^*)$, the best bound for a given \mathbf{x} , could be smaller than the bound c . It should not be surprising that the separation oracle for the dual (DP1) is related to the preceding characterization. The separation oracle for (DP1) is the following: given μ and weight vector \mathbf{y} , normalized such that $\sum_i x_i y_i = 1$, check whether there is any $\phi \in \Phi^*$ such that $\sum_{i \in N} q_{i, \phi} y_i > \mu$ and if so output a separating hyperplane. To see whether there is a violated constraint, it suffices to evaluate $\max_{\phi \in \Phi^*} \sum_{i \in N} q_{i, \phi} y_i$ and compare it with μ . Following the previous discussion, this expression is equal to $\mathbf{E}_{R(\mathbf{x})} \left[\max_{S \subseteq R, S \in \mathcal{I}} \sum_{i \in S} y_i \right]$. One can accurately estimate this quantity as follows. First, we sample a random set R using marginals given by \mathbf{x} . Then we find a maximum \mathbf{y} -weight subset of R that is contained in \mathcal{I} . This gives an unbiased estimator, and to get a high-accuracy estimate we repeat the process sufficiently many times and take the average value. Thus, the algorithmic problem needed for the separation oracle is the maximum weight independent set problem for \mathcal{I} : given

weights \mathbf{y} on N and a $A \subseteq N$ output a maximum weight subset of A in \mathcal{I} . The sampling creates an additive error ϵ in estimating $\mathbf{E}_{R(\mathbf{x})} [\max_{S \subseteq R, S \in \mathcal{I}} \sum_{i \in S} y_i]$ which results in a corresponding loss in finding the optimum solution value μ^* to (DP1). To implement the ellipsoid algorithm we also need to find a separating hyperplane if there is a violated constraint. A natural strategy would be to output the hyperplane corresponding to the violating constraint found while evaluating $\max_{\phi \in \Phi^*} q_{i, \phi} y_i$. However, we do not necessarily have the exact coefficients $q_{i, \phi}$ for the constraint since we use random sampling. We describe in Section C of the appendix the technical details in implementing the ellipsoid algorithm with sufficiently accurate estimates obtained from sampling. For now assume we can find a separating hyperplane corresponding to the most violated constraint. The ellipsoid algorithm can then be used to find a polynomial number of dual constraints that certify that the dual optimum is at least $\mu^* - \epsilon$ where μ^* is the actual dual optimum value. By strong duality $\mu^* = c(\mathbf{x}, \Phi^*)$. We then solve the primal (LP1) by restricting it to the variables that correspond to the dual constraints found by the ellipsoid algorithm. This gives a primal feasible solution of value $c(\mathbf{x}, \Phi^*) - \epsilon$ and this solution is the desired CR scheme. We observe that the primal can be solved efficiently since the number of variables and constraints is polynomial; here too we do not have the precise coefficients $q_{i, \phi}$ but we can use the estimates that come from the dual — see Section C. To summarize, an algorithm for finding a maximum weight independent set in \mathcal{I} , together with sampling and the ellipsoid algorithm, can be used to efficiently find a $(b, c(\mathbf{x}, \Phi^*) - \epsilon)$ -balanced CR scheme where ϵ is an error tolerance; the running time depends polynomially on the input size and $1/\epsilon$. The proof can be easily adapted to show that an α -approximation for the max-weight independent set problem gives a $\alpha \cdot c(\mathbf{x}, \Phi^*) - \epsilon$ CR scheme.

Monotonicity: The discussion so far did not consider the issue of monotonicity. One way to adapt the above approach to monotone schemes is to define Φ to be the family of all deterministic monotone CR schemes and solve (LP1) restricted to Φ . A deterministic scheme ϕ is monotone if it has the property that $i \in \phi(A)$ implies that $i \in \phi(A')$ for all $A' \subset A$. Distributions of deterministic monotone schemes certainly yield a monotone CR scheme. Interestingly, it is not true that all monotone randomized CR schemes can be obtained as distributions of deterministic ones. Now the question is whether we can solve (LP1) restricted to monotone deterministic schemes. In general this is a non-trivial problem. However, the ellipsoid-based algorithm to compute $c(\mathbf{x}, \Phi^*)$ that we described above gives the following important property. In each iteration of the ellipsoid algorithm, the separation oracle uses a maximum-weight independent set algorithm for \mathcal{I} to find a violating constraint; this constraint corresponds to a deterministic scheme ϕ that is obtained by specializing the algorithm to the given weight vector \mathbf{y} . Therefore, if the maximum-weight independent set algorithm is monotone, then all the constraints generated in the ellipsoid algorithm correspond to monotone schemes. Since we solve the primal (LP1) only for the schemes generated by the separation oracle for the dual (DP1), it follows that there is an optimum solution to (LP1) that is a distribution over monotone schemes! In such a case $c(\mathbf{x}, \Phi^*) = c(\mathbf{x}, \Phi)$ and there is no loss in using monotone schemes. For matroids the greedy algorithm to find a maximum weight independent set is a monotone algorithm. Thus, for matroids, the above approach of solving (DP1) and (LP1) can be used to obtain a close to optimal monotone (b, c) -balanced CR scheme. It remains to determine the value of the optimal c and we analyze it in Section 4.4. It may be the case that there is no monotone maximum weight independent set algorithm for some given \mathcal{I} , say the intersection of two matroids. In that case we can use an approximate monotone algorithm instead.

We summarize the above discussions in the following theorem.

Theorem 4.3. *There is a (b, c) -balanced CR scheme for $P_{\mathcal{I}}$ iff $\mathbf{E}_{R(\mathbf{x})} [\max_{S \subseteq R, S \in \mathcal{I}} \sum_{i \in S} y_i] \geq c \sum_i y_i x_i$ for all $\mathbf{x} \in bP_{\mathcal{I}}$ and $\mathbf{y} \geq 0$. Moreover, if there is a polynomial-time deterministic algorithm to find a maximum weight independent set in \mathcal{I} , then for any b and $\epsilon > 0$, there is a randomized efficiently implementable $(b, c^* - \epsilon)$ -balanced CR scheme for $P_{\mathcal{I}}$ where c^* is the smallest value of c such that there is a (b, c) -balanced CR scheme for $P_{\mathcal{I}}$; the running time is polynomial in the input size and $1/\epsilon$. In addition, if the maximum-weight independent set algorithm is monotone, the resulting CR scheme is monotone.*

Before leveraging the above theorem to design close to optimal CR schemes for matroids, we highlight an interesting connection between CR schemes and a concept known as *correlation gap*. This connection is a further insight that we gain through the linear programs (LP1) and (DP1).

4.3 Connection to correlation gap

In this section we highlight a close connection between CR schemes and a concept known as *correlation gap* [2]. The correlation gap is a function on set functions that measures how much the expected value of a set function with respect to some random input can vary, if only the marginal probabilities of the input are fixed. We first show how one can naturally extend this notion to sets $\mathcal{I} \subseteq 2^N$. Then, by exploiting the dual LP formulation of CR schemes (DP1), we present a close relationship of the notion of correlation gap, interpreted in terms of constraints, and the existence of strong CR schemes.

Definition 4.4. For a set function $f : 2^N \rightarrow \mathbb{R}_+$, the correlation gap is defined as

$$\kappa(f) = \inf_{\mathbf{x} \in [0,1]^N} \frac{\mathbf{E}[f(R(\mathbf{x}))]}{f^+(\mathbf{x})},$$

where $R(\mathbf{x})$ is a random set independently containing each element i with probability x_i , and

$$f^+(\mathbf{x}) = \max\left\{\sum_S \alpha_S f(S) : \sum_S \alpha_S \mathbf{1}_S = \mathbf{x}, \sum_S \alpha_S = 1, \alpha_S \geq 0\right\}$$

is the maximum possible expectation of f over distributions with expectation \mathbf{x} . Furthermore, for a class of functions \mathcal{C} , the correlation gap is defined by $\kappa(\mathcal{C}) = \inf_{f \in \mathcal{C}} \kappa(f)$.

In other words, the correlation gap is the worst-case ratio between the multilinear extension $F(\mathbf{x}) = \mathbf{E}[f(R(\mathbf{x}))]$ and the concave closure $f^+(\mathbf{x})$. We remark that we define the correlation gap as a number $\kappa \in [0, 1]$, to be in line with the parameter c in our notion of a (b, c) -balanced CR scheme (the higher the better). The definition in [2] uses the inverse ratio.

The relationship between CR schemes and correlation gap arises as follows.

Definition 4.5. For $\mathcal{I} \subseteq 2^N$, we define the correlation gap as $\kappa(\mathcal{I}) = \inf_{\mathbf{x} \in P_{\mathcal{I}}, \mathbf{y} \geq 0} \frac{1}{\sum_i x_i y_i} \mathbf{E}[\max_{S \subseteq \mathcal{I}} \sum_{i \in S} y_i]$, where $R = R(\mathbf{x})$ contains element i independently with probability x_i .

The reason we call this quantity a correlation gap (considering Definition 4.4), is that this quantity is equal to the correlation gap of the *weighted rank function* corresponding to \mathcal{I} (see Lemma 4.7 below).

Theorem 4.6. The correlation gap of \mathcal{I} is equal to the maximum c such that \mathcal{I} admits a c -balanced CR scheme.

Proof. The correlation gap of \mathcal{I} is equal to the optimum value of (DP1). By LP duality, this is equal to the optimum of the primal (LP1), which is the best value of c for which there is a c -balanced CR scheme. \square

The following lemma shows a close connection between the correlation gap of a solution set \mathcal{I} and the correlation gap of the respective rank function. More precisely, the correlation gap of \mathcal{I} corresponds to the worst (i.e. smallest) correlation gap of the respective rank function over all weight vectors.

Lemma 4.7. For $\mathcal{I} \subseteq 2^N$ and weight vector $\mathbf{y} \geq 0$, let $r_{\mathbf{y}}(R) = \max_{S \subseteq \mathcal{I}} \sum_{i \in S} y_i$ denote the associated *weighted rank function*. Then $\kappa(\mathcal{I}) = \inf_{\mathbf{y} \geq 0} \kappa(r_{\mathbf{y}})$.

Proof. Using the notation $r_{\mathbf{y}}(R)$ for the weighted rank function with weights \mathbf{y} , the correlation gap of \mathcal{I} can be rewritten as $\kappa(\mathcal{I}) = \inf_{\mathbf{x} \in P_{\mathcal{I}}, \mathbf{y} \geq 0} \frac{\mathbf{E}[r_{\mathbf{y}}(R(\mathbf{x}))]}{\sum_i x_i y_i}$, where $R(\mathbf{x})$ contains elements independently with probabilities x_i . We first observe that for any $\mathbf{x} \in P_{\mathcal{I}}$, we have $r_{\mathbf{y}}^+(\mathbf{x}) = \sum_i x_i y_i$. Hence, let $\mathbf{x} \in P_{\mathcal{I}}$, and consider a convex combination $\mathbf{x} = \sum_{S \in \mathcal{I}} \alpha_S \mathbf{1}_S$, $\sum \alpha_S = 1$, $\alpha_S \geq 0$ with $r_{\mathbf{y}}^+(\mathbf{x}) = \sum_{S \in \mathcal{I}} \alpha_S y(S)$. Since the weighted rank function of a feasible set $S \in \mathcal{I}$ is simply its weight we obtain

$$r_{\mathbf{y}}^+(\mathbf{x}) = \sum_{S \in \mathcal{I}} \alpha_S y(S) = \mathbf{y} \cdot \sum_{S \in \mathcal{I}} \alpha_S \mathbf{1}_S = \mathbf{y} \cdot \mathbf{x} = \sum_i x_i y_i,$$

as claimed. Therefore,

$$\kappa(\mathcal{I}) = \inf_{\mathbf{x} \in P_{\mathcal{I}}, \mathbf{y} \geq 0} \frac{\mathbf{E}[r_{\mathbf{y}}(R(\mathbf{x}))]}{\sum_i x_i y_i} = \inf_{\mathbf{x} \in P_{\mathcal{I}}, \mathbf{y} \geq 0} \frac{\mathbf{E}[r_{\mathbf{y}}(R(\mathbf{x}))]}{r_{\mathbf{y}}^+(\mathbf{x})}.$$

To prove the claim it remains to show that

$$\inf_{\mathbf{x} \in P_{\mathcal{I}}, \mathbf{y} \geq 0} \frac{\mathbf{E}[r_{\mathbf{y}}(R(\mathbf{x}))]}{r_{\mathbf{y}}^+(\mathbf{x})} = \inf_{\mathbf{x} \in [0,1]^N, \mathbf{y} \geq 0} \frac{\mathbf{E}[r_{\mathbf{y}}(R(\mathbf{x}))]}{r_{\mathbf{y}}^+(\mathbf{x})}. \quad (12)$$

Let $\mathbf{y} \geq 0$. We will prove (12) by showing that for any point $\mathbf{x} \in [0,1]^N$ there is a point $\mathbf{x}' \in P_{\mathcal{I}}$ with $\mathbf{x}' \leq \mathbf{x}$ (coordinate-wise), and satisfying $r_{\mathbf{y}}^+(\mathbf{x}') \geq r_{\mathbf{y}}^+(\mathbf{x})$. Since $r_{\mathbf{y}}$ is monotone, we then obtain $\mathbf{E}[r_{\mathbf{y}}(R(\mathbf{x}))]/r_{\mathbf{y}}^+(\mathbf{x}) \geq \mathbf{E}[r_{\mathbf{y}}(R(\mathbf{x}))]/r_{\mathbf{y}}^+(\mathbf{x}')$, showing that the infimum over \mathbf{x} on the right-hand side of (12) can indeed be restricted to the polytope $P_{\mathcal{I}}$. Let $\mathbf{x} = \sum_{S \subseteq N} \alpha_S \mathbf{1}_S$, $\sum_{S \subseteq N} \alpha_S = 1$, $\alpha_S \geq 0$ be a convex combination of \mathbf{x} such that $r_{\mathbf{y}}^+(\mathbf{x}) = \sum_{S \subseteq N} \alpha_S r_{\mathbf{y}}(S)$. For every $S \subseteq N$, let $I(S) \subseteq S$ be a maximum weight independent set, hence $r_{\mathbf{y}}(S) = y(I(S))$. The point $\mathbf{x}' = \sum_{S \subseteq N} \alpha_S \mathbf{1}_{I(S)}$ clearly satisfies $\mathbf{x}' \leq \mathbf{x}$, and furthermore

$$r_{\mathbf{y}}^+(\mathbf{x}') \geq \sum_{S \in \mathcal{I}} \left(\sum_{W \subseteq N, I(W)=S} \alpha_W \right) r_{\mathbf{y}}(S) = \sum_{S \subseteq N} \alpha_S r_{\mathbf{y}}(S) = r_{\mathbf{y}}^+(\mathbf{x}).$$

□

4.4 Contention resolution for matroids

In this section we prove the following theorem on CR schemes for matroids.

Theorem 4.8. *For any matroid $\mathcal{M} = (N, \mathcal{I})$ on n elements there exists a $\left(b, \frac{1-(1-\frac{b}{n})^n}{b}\right)$ -balanced CR scheme for the polytope $P(\mathcal{M})$.*

We later address monotonicity of the scheme and constructive aspects. To prove Theorem 4.8 we rely on the characterization formalized in Theorem 4.3. It suffices to prove for $\mathbf{x} \in b \cdot P_{\mathcal{I}}$ and any non-negative weight vector $\mathbf{y} \geq 0$ that $\mathbf{E}_R [\max_{S \subseteq R, S \in \mathcal{I}} \sum_{i \in S} y_i] \geq c \sum_{i \in N} x_i y_i$, with $c = \frac{1-(1-\frac{b}{n})^n}{b}$ where R contains each $i \in N$ independently with probability x_i and $\mathbf{x} \in b \cdot P_{\mathcal{I}}$. For a given weight vector $\mathbf{y} \geq 0$ on N and a set $S \subseteq N$ let $r_{\mathbf{y}}(S)$ denote the weight of a maximum weight independent set contained in S ; in other words $r_{\mathbf{y}}$ is the weighted rank function of the matroid \mathcal{M} . Restating, it remains to prove

$$\mathbf{E}[r_{\mathbf{y}}(R)] \geq \frac{1 - (1 - \frac{b}{n})^n}{b} \sum_{i \in N} y_i x_i, \quad (13)$$

It is well-known that a simple greedy algorithm can be used to compute $r_{\mathbf{y}}(S)$ (in fact an independent set $S' \subseteq S$ of maximum weight with respect to y_i): Start with $S' = \emptyset$, consider the elements of S in non-increasing order of their weight y_i and add the current element i to S' if $S' + i$ is independent, otherwise discard i .

To show (13), which is a general property of weighted matroid rank functions, we prove a more general result that holds for any non-negative monotone submodular function. The main ingredient for this is a lower bound on the multilinear extension, which is stated in Lemma 4.10. A slightly weaker form of Lemma 4.10, which we state as Lemma 4.9, will be presented first, due to its concise proof. The proof of Lemma 4.10 is deferred to the appendix. Both lemmas can be seen as an extension of the property that the correlation gap for monotone submodular functions is $1 - 1/e$ [6].

Lemma 4.9. *If $f : 2^N \rightarrow \mathbb{R}_+$ is a monotone submodular function, $F : [0,1]^N \rightarrow \mathbb{R}_+$ its multilinear extension, and $f^+ : [0,1]^N \rightarrow \mathbb{R}_+$ its concave closure, then for any $b \in [0,1]$ and $\mathbf{p} \in [0,1]^N$,*

$$F(b \cdot \mathbf{p}) \geq (1 - e^{-b}) f^+(\mathbf{p}).$$

Proof. We use another extension of a monotone submodular function, defined in [6]:

$$f^*(\mathbf{p}) = \min_S \left(f(S) + \sum_i p_i f_S(i) \right).$$

It is shown in [6] that $f^*(\mathbf{p}) \geq f^+(\mathbf{p})$ for all $\mathbf{p} \in [0, 1]^N$. Consider the function $\phi(t) = F(t\mathbf{p})$ for $t \in [0, 1]$, i.e. the multilinear extension on the line segment between $\mathbf{0}$ and \mathbf{p} . We prove that $\phi(t)$ satisfies a differential equation similar to the analysis of the continuous greedy algorithm [7], which leads immediately to the statement of the lemma. We have

$$\frac{d\phi}{dt} = \mathbf{p} \cdot \nabla F(t\mathbf{p}) = \sum_i p_i \frac{\partial F}{\partial x_i} \Big|_{\mathbf{x}=t\mathbf{p}}.$$

By properties of the multilinear extension, we have $\frac{\partial F}{\partial x_i} \Big|_{\mathbf{x}=t\mathbf{p}} = \mathbf{E}[f(R+i) - f(R-i)] \geq \mathbf{E}[f_R(i)]$, where R is a random set sampled independently with probabilities $x_i = tp_i$ (see [7] for more details). Therefore,

$$\frac{d\phi}{dt} = \sum_i p_i \frac{\partial F}{\partial x_i} \Big|_{\mathbf{x}=t\mathbf{p}} \geq \sum_i p_i \mathbf{E}[f_R(i)] = \mathbf{E}[\sum_i p_i f_R(i)] \geq \mathbf{E}[f^*(\mathbf{p}) - f(R)]$$

by the definition of $f^*(\mathbf{p})$. Finally, $\mathbf{E}[f(R)] = F(t\mathbf{p}) = \phi(t)$, hence we obtain the following differential inequality:

$$\frac{d\phi}{dt} \geq f^*(\mathbf{p}) - \phi(t)$$

under the initial condition $\phi(0) \geq 0$. We solve this as follows: $\frac{d}{dt}(e^t \phi(t)) = e^t \phi(t) + e^t \frac{d\phi}{dt} \geq e^t f^*(\mathbf{p})$ which implies that

$$e^b \phi(b) \geq e^0 \phi(0) + \int_0^b e^t f^*(\mathbf{p}) dt \geq (e^b - 1) f^*(\mathbf{p}).$$

Considering that $\phi(b) = F(b\mathbf{p})$ and $f^*(\mathbf{p}) \geq f^+(\mathbf{p})$, this proves the lemma. \square

A more fine-grained analysis leads to the following strengthened version of Lemma 4.9, whose proof can be found in Appendix B.

Lemma 4.10. *If $f : 2^N \rightarrow \mathbb{R}_+$ is a monotone submodular function, $F : [0, 1]^N \rightarrow \mathbb{R}_+$ its multilinear extension, and $f^+ : [0, 1]^N \rightarrow \mathbb{R}_+$ its concave closure, then for any $b \in [0, 1]$ and $\mathbf{p} \in [0, 1]^N$,*

$$F(b \cdot \mathbf{p}) \geq \left(1 - \left(1 - \frac{b}{n}\right)^n\right) f^+(\mathbf{p}).$$

Lemma 4.10 implies (13), and therefore completes the proof of Theorem 4.8, by setting $f = r_{\mathbf{y}}$ and $b \cdot \mathbf{p} = \mathbf{x}$. Notice that the multilinear extension of $r_{\mathbf{y}}$ evaluated at \mathbf{x} is $\mathbf{E}[r_{\mathbf{y}}(R)]$. Furthermore, $r_{\mathbf{y}}(\mathbf{p}) = \sum_{i \in N} y_i p_i = \sum_{i \in N} y_i \frac{x_i}{b}$ if \mathbf{p} is in the matroid polytope. Hence we obtain (13):

$$\begin{aligned} \mathbf{E}[r_{\mathbf{y}}(R)] &\geq \left(1 - \left(1 - \frac{b}{n}\right)^n\right) r_{\mathbf{y}}^+(\mathbf{p}) = \left(1 - \left(1 - \frac{b}{n}\right)^n\right) \sum_{i \in N} y_i \frac{x_i}{b} \\ &= \frac{1 - \left(1 - \frac{b}{n}\right)^n}{b} \sum_{i \in N} y_i x_i. \end{aligned}$$

Theorem 4.3 also shows that an efficient algorithm for computing $r_{\mathbf{y}}$ results in an efficiently implementable near-optimal CR scheme. It is well-known that a simple greedy algorithm can be used to compute $r_{\mathbf{y}}(S)$ (in fact an independent set $S' \subseteq S$ of maximum weight): Start with $S' = \emptyset$, consider the elements of S in non-increasing order of their weight and add the current element i to S' if $S' + i$ is independent, otherwise discard i . Moreover, it is easy to see that this algorithm is monotone — the ordering of the elements by weight does not depend on the set S and hence if an element i is included when evaluating $r_{\mathbf{y}}(A)$ then it will be included in evaluating $r_{\mathbf{y}}(B)$ for any $B \subset A$. We thus obtain our main result for CR schemes in the context of matroids by combining Theorem 4.3 for a choice of ϵ satisfying $\epsilon \leq \frac{b}{10n}$ with Theorem 4.8, and by using the inequality $(1 - \frac{b}{n})^n \leq e^{-b} - \frac{b^2}{10n}$ ¹¹.

¹¹This inequality can be obtained by observing that $1 - x + \frac{x^2}{3} \leq e^{-x}$ for $x \in [0, 1]$, and hence $(1 - \frac{b}{n})^n \leq (e^{-\frac{b}{n}} - \frac{b^2}{3n^2})^n$. Let $y = e^{-\frac{b}{n}}$ and $z = \frac{b^2}{3n^2}$ for simplicity. One can easily check that for these values of y and z we have $(y - z)^n \leq y^n - ny^{n-1}z + \frac{n^2}{2}y^{n-2}z^2$. Expanding the last expression and using $n \geq 2$, since the inequality is trivially true for $n = 1$, the desired inequality follows.

Corollary 4.11. For any matroid \mathcal{M} , and $\mathbf{x} \in b \cdot P_{\mathcal{I}}$, there is an efficiently implementable $(b, \frac{1-e^{-b}}{b})$ -balanced and monotone CR scheme.

As shown by the following theorem, the CR schemes that can be obtained according to Corollary 4.11 are, up to an additive ε , asymptotically optimal.

Theorem 4.12. For any $b \in (0, 1]$, there is no (b, c) -balanced CR scheme for uniform matroids of rank one on n elements with $c > \frac{1-(1-\frac{b}{n})^n}{b}$.

Proof. Let $\mathcal{M} = (N, \mathcal{I})$ be the uniform matroid of rank 1 over $n = |N|$ elements, and consider the point $\mathbf{x} \in b \cdot P_{\mathcal{I}}$ given by $x_i = b/n$ for $i \in N$. Let R be a random set containing each element $i \in N$ independently with probability x_i . The expected rank of R is given by

$$\mathbf{E}[r(R)] = 1 - \Pr[R = \emptyset] = 1 - \left(1 - \frac{b}{n}\right)^n. \quad (14)$$

Moreover, any (b, c) -balanced CR scheme returning a set $I \in \mathcal{I}$ satisfies

$$\mathbf{E}[|I|] = \sum_{i \in N} \Pr[i \in I] \geq \sum_{i \in N} \frac{bc}{n} = bc. \quad (15)$$

Since I is an independent subset of R we have $\mathbf{E}[r(R)] \geq \mathbf{E}[|I|]$, and the claim follows by (14) and (15). \square

A simple $(b, 1-b)$ -balanced CR scheme: Here we describe a sub-optimal $(b, 1-b)$ -balanced CR scheme for matroid polytopes. Its advantage is that it is *deterministic*, simpler and computationally less expensive than the optimal scheme that requires solving a linear program. Moreover, Lemma 4.13 that is at the heart of the scheme, is of independent interest and may find other applications. A similar lemma was independently shown in [11] (prior to our work but in a different context). Let $\mathcal{M} = (N, \mathcal{I})$ be a matroid. For $S \subseteq N$ recall that $r(S)$ is the rank of S in \mathcal{M} . The span of a set S denoted by $\text{span}(S)$ is the set of all elements $i \in N$ such that $r(S+i) = r(S)$.

Lemma 4.13. If $\mathcal{M} = (N, \mathcal{I})$ is a matroid, $\mathbf{x} \in P(\mathcal{M})$, $b \in [0, 1]$ and R a random set such that $\Pr[i \in R] = bx_i$, then there is an element i_0 such that $\Pr[i_0 \in \text{span}(R)] \leq b$.

Proof. Let $r(S) = \max\{|I| : I \subseteq S \text{ \& } I \in \mathcal{I}\}$ denote the rank function of matroid $\mathcal{M} = (N, \mathcal{I})$. Since $\mathbf{x} \in P(\mathcal{M})$, it satisfies the rank constraints $x(S) \leq r(S)$. For $S = \text{span}(R)$, we get

$$x(\text{span}(R)) \leq r(\text{span}(R)) = r(R) \leq |R|.$$

Recall that R is a random set where $\Pr[i \in R] = bx_i$. We take the expectation on both sides:

$$\mathbf{E}[x(\text{span}(R))] = \sum_i x_i \Pr[i \in \text{span}(R)], \text{ and } \mathbf{E}[|R|] = \sum_i \Pr[i \in R] = b \sum_i x_i.$$

Therefore,

$$\sum_{i \in N} x_i \Pr[i \in \text{span}(R)] \leq b \sum_{i \in N} x_i.$$

This implies that there must be an element i_0 such that $\Pr[i_0 \in \text{span}(R)] \leq b$. \square

We remark that the inequality $\sum_i x_i \Pr[i \in \text{span}(R)] \leq \mathbf{E}[|R|]$ has an interesting interpretation: If $\mathbf{x} \in P(\mathcal{M})$, we sample R with probabilities x_i , then let $S = \text{span}(R)$ and sample again $S' \subseteq S$ with probabilities x_i , then $\mathbf{E}[|S'|] \leq \mathbf{E}[|R|]$. We do not use this in the following, though.

Theorem 4.14. For any matroid \mathcal{M} and any $b \in [0, 1]$, there is a deterministic $(b, 1-b)$ -balanced CR scheme.

Proof. Let $\mathbf{x} \in P(\mathcal{M})$ and sample R with probabilities bx_i . We define an ordering of elements as follows. By Lemma 4.13, there is an element i_0 such that $\Pr[i \in \text{span}(R)] \leq b$. We place i_0 at the end of the order. Then, since \mathbf{x} restricted to $N \setminus \{i_0\}$ is in the matroid polytope of $\mathcal{M} \setminus \{i_0\}$, we can recursively find an ordering by the same rule. If the elements are labeled $1, 2, \dots, |N|$ in this order, we obtain that $\Pr[i \in \text{span}(R \cap [i])] \leq b$ for every i . In fact, we are interested in the event that i is in the span of the preceding elements, $R \cap [i-1]$. This is a subset of $R \cap [i]$, and hence

$$\Pr[i \in \text{span}(R \cap [i-1])] \leq \Pr[i \in \text{span}(R \cap [i])] \leq b.$$

The CR scheme is as follows:

- Sample R with probabilities bx_i .
- For each element i , if $i \in R \setminus \text{span}(R \cap [i-1])$, then include it in I .

Obviously, $r(I \cap [i]) = r(I \cap [i-1]) + 1$ whenever $i \in I$, so $r(I) = |I|$ and I is an independent set.

To bound the probability of appearance of i , observe that the appearance of elements in $[i-1]$ is independent of the appearance of i itself, and hence the events $i \in R$ and $i \notin \text{span}(R \cap [i-1])$ are independent. As we argued, $\Pr[i \in \text{span}(R \cap [i-1])] \leq b$. We conclude:

$$\Pr[i \in I \mid i \in R] = \Pr[i \notin \text{span}(R \cap [i-1])] \geq 1 - b.$$

□

To implement the scheme we need to make Lemma 4.13 algorithmic. We can accomplish it by random sampling. Fix an element i . Pick a random set R and check if $i \in \text{span}(R)$; repeat sufficiently many times to obtain an accurate estimate of $\Pr[i \in \text{span}(R)]$. We note that although the scheme itself is deterministic once we find an ordering of the elements, the construction of the ordering is randomized due to the estimation of $\Pr[i \in \text{span}(R)]$ via sampling.

4.5 Contention resolution for knapsacks

Here we sketch a contention resolution scheme for knapsack constraints. This essentially follows from known techniques; we remark that Kulik, Shachnai and Tamir [34, 35] showed how to round a fractional solution to the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ for any constant number of knapsack constraints and any non-negative submodular function, while losing a $(1 - \varepsilon)$ factor for an arbitrarily small $\varepsilon > 0$. Our goal is to show that these techniques can be implemented in a black-box fashion and integrated in our framework.

Let $N = \{1, 2, \dots, n\}$ and let $a_1, a_2, \dots, a_n \in [0, 1]$ be sizes of the n items. The independence system induced by a single knapsack constraint is $\mathcal{F} = \{S : \sum_{i \in S} a_i \leq 1\}$ and its natural relaxation has a variable x_i for $1 \leq i \leq n$ and is defined as $P_{\mathcal{F}} = \{\mathbf{x} \in [0, 1]^n : \sum_i a_i x_i \leq 1\}$. We refer to this as the knapsack polytope.

We prove the following lemma.

Lemma 4.15. *For any $b \in (0, 1/2)$ there is a monotone $(b, 1 - 2b)$ -balanced CR scheme for the knapsack polytope. If, for some $\delta \in (0, \frac{1}{2})$, $a_i \leq \delta$ for $1 \leq i \leq n$, then for any $b \in (0, \frac{1}{2e})$ there is a monotone $(b, 1 - (2eb)^{(1-\delta)/\delta})$ -balanced CR scheme. Further, for any $0 < 2\delta < \varepsilon < \frac{1}{2}$, if $a_i \leq \delta$ for $1 \leq i \leq n$ then there is a monotone $(1 - \varepsilon, 1 - e^{-\Omega(\varepsilon^2/\delta)})$ -balanced CR scheme.*

Proof. The CR scheme is the same for all the cases and works as follows: given $\mathbf{x} \in b \cdot P_{\mathcal{F}}$ we sample R with probabilities x_i . To obtain I from R we sort the items from R in an order of decreasing size and set I to be the largest prefix of this sequence that fits in the knapsack. Equivalently, we consider the items from R in an order of decreasing size and add the current item to I if it maintains feasibility in the knapsack, else we discard it. It is easy to see that this scheme is monotone.

First, we consider the general case where there are no restrictions on the item sizes. Let $N_{\text{big}} = \{i \in N \mid a_i > 1/2\}$ be the big items in N and let $N_{\text{small}} = N \setminus N_{\text{big}}$ be the small items. The probability of at least one big element being in R is at most $2b$ since

$$\Pr[N_{\text{big}} \cap R \neq \emptyset] \leq \sum_{i \in N_{\text{big}}} x_i \leq 2 \sum_{i \in N_{\text{big}}} a_i x_i \leq 2b.$$

The first inequality is via the union bound, the second inequality is using the fact that $a_i > 1/2$ for all big items, and the third inequality follows from $\mathbf{x} \in b \cdot \mathcal{P}_{\mathcal{F}}$. Thus $\Pr[N_{\text{big}} \cap R = \emptyset] \geq 1 - 2b$.

Fix some $j \in N$. We need to lower bound $\Pr[j \in I \mid j \in R]$ where I is the output of the CR scheme we described. First consider the case that j is big. Since all big items are considered before any small item, j is accepted if it is the unique big item in R . Since items are included in R independently, we have

$$\Pr[j \in I \mid j \in R] \geq \Pr[(N_{\text{big}} \setminus \{j\}) \cap R = \emptyset \mid j \in R] = \Pr[(N_{\text{big}} \setminus \{j\}) \cap R = \emptyset] \geq \Pr[N_{\text{big}} \cap R = \emptyset] \geq 1 - 2b.$$

Now we consider the case that j is small. Since $a_j \leq 1/2$, j will be accepted if $a(R \setminus \{j\}) \leq 1/2$. For any $S \subseteq N$, we have $\mathbf{E}[a(S \cap R)] = \sum_{i \in S} a_i \Pr[i \in R] \leq \sum_{i \in S} a_i x_i$. In particular $\mathbf{E}[a(R)] \leq \sum_{i \in N} a_i x_i \leq b$. By Markov's inequality, $\Pr[a(R \setminus \{j\}) > 1/2] \leq \Pr[a(R) > 1/2] \leq 2b$. Therefore each small item is accepted with probability at least $1 - 2b$. The same analysis holds for a simpler CR scheme based on an ordering of elements in R in which all big items are considered before any small item.

Now we consider the case that for $1 \leq i \leq n$, $a_i \leq \delta$ for some parameter $\delta \leq 1/2$. Fix some $j \in N$. It is clear that $j \in I$ if $j \in R$ and $\sum_{i \in R \setminus \{j\}} a_i \leq 1 - \delta$. Conditioned on $j \in R$, the probability of this event is at least $1 - \Pr[\sum_{i \in R} a_i > 1 - \delta]$. We upper bound $\Pr[\sum_{i \in R} a_i > 1 - \delta]$ via Chernoff bounds. Let Y_i be the indicator random variable for i to be chosen in R ; $\Pr[Y_i = 1] = x_i$. Let $Y = \sum_{i \in R} a_i Y_i$. We have $\mathbf{E}[Y] = \sum_{i \in R} a_i x_i \leq b < 1 - \delta$ by feasibility of \mathbf{x} . We are interested in $\Pr[Y > 1 - \delta] = \Pr[\sum_{i \in R} a_i > 1 - \delta]$. We can assume that $\mathbf{E}[Y] = b$ by adding dummy elements if necessary; this can only increase $\Pr[Y > 1 - \delta]$.

We use the standard Chernoff bound, $\Pr[Z > (1 + \alpha)\mu] \leq (e^\alpha / (1 + \alpha)^{1+\alpha})^\mu$ where Z is a sum of random variables in $[0, 1]$ and $\mu = \mathbf{E}[Z]$. To apply this bound to our setting, we consider $Z = Y/\delta = \sum_{i \in R} a_i Y_i / \delta = \sum_{i \in R} Z_i$ where $Z_i = a_i Y_i / \delta$ is a random variable in $[0, 1]$ since $a_i \leq \delta$. Thus, $\Pr[Y > 1 - \delta] = \Pr[Z > (1 + \alpha)\mathbf{E}[Z]] \leq (e^\alpha / (1 + \alpha)^{1+\alpha})^\mu$ where $\mu = \mathbf{E}[Z] = b/\delta$ and $(1 + \alpha) = (1 - \delta)/b$. Using $\delta \leq \frac{1}{2}$, we obtain that

$$\Pr[Y > 1 - \delta] \leq \left(\frac{e}{1 + \alpha} \right)^{(1+\alpha)\mu} \leq \left(\frac{eb}{1 - \delta} \right)^{(1-\delta)/\delta} \leq (2eb)^{(1-\delta)/\delta}.$$

Finally, let's consider the case where $b = 1 - \epsilon$ and $a_i \leq \delta \leq \frac{\epsilon}{2} \leq \frac{1}{4}$ for all i . Here we use the Chernoff-Hoeffding bound $\Pr[Z > (1 + \alpha)\mu] < e^{-\alpha^2 \mu / 3}$ for $\alpha \in (0, 1)$ and Z being a sum of random variables bounded by $[0, 1]$. We estimate the probability that conditioned on $j \in R$, all of R fits in the knapsack. Since $a_j \leq \delta \leq \frac{\epsilon}{2}$, this probability is

$$\Pr \left[\sum_{i \in R} a_i \leq 1 \mid j \in R \right] \geq \Pr \left[\sum_{i \in R \setminus \{j\}} a_i \leq 1 - \epsilon/2 \right].$$

We have $\mathbf{E}[\sum_{i \in R \setminus \{j\}} a_i] = \sum_{i \in N \setminus \{j\}} a_i x_i \leq 1 - \epsilon$. We can in fact assume that $\mu = \mathbf{E}[\sum_{i \in R \setminus \{j\}} a_i] = 1 - \epsilon$, by adding dummy elements that can only increase the probability of overflowing $1 - \epsilon/2$. Applying the Chernoff bound for random variables bounded by δ (after rescaling as above), we obtain

$$\Pr \left[\sum_{i \in R \setminus \{j\}} a_i > 1 - \epsilon/2 \right] \leq \Pr \left[\sum_{i \in R \setminus \{j\}} a_i > (1 + \epsilon/2)\mu \right] \leq e^{-\epsilon^2 \mu / (12\delta)} = e^{-\Omega(\epsilon^2/\delta)}.$$

□

The $(1 - \epsilon, 1 - e^{-\Omega(\epsilon^2/\delta)})$ -balanced CR scheme is directly applicable only if the item sizes are relatively small compared to the knapsack capacity. However, standard enumeration tricks allow us to apply this scheme to general instances as well. This can be done for any constant number of knapsack constraints. We formulate this as follows.

Corollary 4.16. *For any constant $k \geq 1$ and $\epsilon > 0$, there is a constant n_0 (that depends only on ϵ) such that for any submodular maximization instance involving k knapsack constraints (and possibly other constraints), there is a set T of at most n_0 elements and a residual instance on the remaining elements such that*

- Any α -approximate solution to the residual instance together with T is an $\alpha(1 - k\epsilon)$ -approximate solution to the original instance.

- In the residual instance, each knapsack constraint admits a $(1 - \varepsilon, 1 - \varepsilon)$ -balanced CR scheme.

Proof. Given $\varepsilon > 0$, let $\delta = O(\varepsilon^2 / \log(1/\varepsilon))$ and $n_0 = 1/(\delta\varepsilon)$. Select T greedily from the optimal solution, by picking elements as long as their marginal contribution is at least $\delta\varepsilon\text{OPT}$; note that $|T| \leq n_0$. We define the residual instance so that S is feasible in the residual instance iff $S \cup T$ is feasible in the original instance. The objective function in the new instance is g defined by setting $g(S) = f(S \cup T)$ for each set $S \subseteq N \setminus T$; note that g is a non-negative submodular function if f is. In addition, in the residual instance we remove all elements whose size for some knapsack constraint is more than $\delta \cdot r$ where r is the residual capacity. The number of such elements in a knapsack can be at most $1/\delta$ and hence they can contribute at most εOPT ; we forgo this value for each knapsack. We obtain a residual instance where all sizes are at most δ with the capacities normalized to 1. By Lemma 4.15, each knapsack admits a $(1 - \varepsilon, 1 - e^{-\Omega(\varepsilon^2/\delta)}) = (1 - \varepsilon, 1 - \varepsilon)$ -balanced CRS. \square

An advantage of this black box approach is that knapsack constraints can be combined arbitrarily with other types of constraints. They do not affect the approximation ratio significantly. However, the enumeration stage affects the running time by an $O(n^{n_0})$ factor.

4.6 Sparse packing systems

We now consider packing constraints of the type $A\mathbf{x} \leq \mathbf{b}$, where $\mathbf{x} \in \{0, 1\}^N$ is the indicator vector of a solution. We can assume without loss of generality that the right-hand side is $\mathbf{b} = \mathbf{1}$. We say that the system is k -sparse, if each column of A has at most k nonzero entries (i.e., each element participates in at most k linear constraints). The approximation algorithms in [4] can be seen to give a contention resolution scheme for k -sparse packing systems.

CR scheme for k -sparse packing systems:

- We say that element j participates in constraint i , if $a_{ij} > 0$. We call an element j *big* for this constraint, if $a_{ij} > 1/2$. Otherwise we call element j *small* for this constraint.
- Sample R with probabilities x_i .
- For each constraint i : if there is exactly one big element in R that participates in i , mark all the small elements in R for this constraint for deletion; otherwise check whether $\sum_{j \in R} a_{ij} > 1$ and if so, mark all elements participating in i for deletion.
- Define I to be R minus the elements marked for deletion.

Based on the analysis in [4], we obtain the following.

Lemma 4.17. *For any $b \in (0, \frac{1}{2k})$, the above is a monotone $(b, 1 - 2kb)$ -balanced CR scheme for k -sparse packing systems.*

Proof. Let $\mathbf{x} = b \cdot \mathbf{y}$ with $\mathbf{y} \in [0, 1]^N$, $A\mathbf{y} \leq \mathbf{1}$. Consider a fixed element j^* . It appears in R with probability x_{j^*} . We analyze the probability that it is removed due to some constraint where it participates. First, note that whether big or small, element j^* cannot be removed due to a constraint i if the remaining elements have size less than $1/2$, i.e. if $\sum_{j \in R \setminus \{j^*\}} a_{ij} < 1/2$. This is because in this case, there is no other big element participating in i , and element j^* is either big in which case it survives, or it is small and then $\sum_{j \in R} a_{ij} \leq 1$, i.e. the constraint is satisfied.

Thus it remains to analyze the event $\sum_{j \in R \setminus \{j^*\}} a_{ij} \geq 1/2$. Note that this is independent of item j^* appearing in R . By the feasibility of $\frac{1}{b}\mathbf{x}$, $\mathbf{E}[\sum_{j \in R \setminus \{j^*\}} a_{ij}] = \sum_{j \neq j^*} x_j a_{ij} \leq b$. By Markov's inequality, $\Pr[\sum_{j \in R \setminus \{j^*\}} a_{ij} \geq 1/2] \leq 2b$. So an element is removed with probability at most $2b$ for each constraint where it participates. By the union bound, it is removed by probability at most $2kb$. \square

Recall the notion of width for a packing system: $W = \lfloor \frac{1}{\max_{i,j} a_{ij}} \rfloor$, where a_{ij} are the entries of the packing matrix (recall that we normalize the right-hand side to be $\mathbf{b} = \mathbf{1}$). Assuming that $W \geq 2$, one can use a simpler CR scheme and improve the parameters.

CR scheme for k -sparse packing systems of width W :

- Sample R with probabilities x_i .
- For each constraint i for which $\sum_{j \in R} a_{ij} > 1$, mark all elements participating in i for deletion.
- Define I to be R minus the elements marked for deletion.

Lemma 4.18. *For any $b \in (0, \frac{1}{2e})$, the above is a monotone $(b, 1 - k(2eb)^{W-1})$ -balanced CR scheme for any k -sparse system of packing constraints of width $W \geq 2$.*

Proof. Again, let $\mathbf{x} = b\mathbf{y}$ with $\mathbf{y} \in [0, 1]^N$, $A\mathbf{y} \leq \mathbf{1}$. Let us consider an element j' and a constraint i that j' participates in. If we condition on j' being present in R , we have $\mu_i = \mathbf{E}[\sum_{j \in R \setminus \{j'\}} a_{ij} \mid j' \in R] = \sum_{j \neq j'} a_{ij} x_{ij} \leq b$. By the width property, we have $a_{ij'} \leq 1/W \leq 1/2$. We use the Chernoff bound for a sum X of independent $[0, 1]$ random variables with $\mu = \mathbf{E}[X]$: $\Pr[X > (1 + \delta)\mu] \leq (e^\delta / (1 + \delta)^{1 + \delta})^\mu \leq (e / (1 + \delta))^{(1 + \delta)\mu}$, with $1 + \delta = (1 - a_{ij'}) / \mu_i \geq 1 / (2b)$. Since our random variables are bounded by $[0, \max a_{ij}]$, we obtain by scaling

$$\begin{aligned} \Pr \left[\sum_{j \in R} a_{ij} > 1 \mid j' \in R \right] &= \Pr \left[\sum_{j \in R \setminus \{j'\}} a_{ij} > 1 - a_{ij'} \right] \\ &\leq \left(\frac{e}{1 + \delta} \right)^{(1 + \delta)\mu_i / \max a_{ij}} \leq (2eb)^{(1 - a_{ij'}) / \max a_{ij}} \leq (2eb)^{W-1}. \end{aligned}$$

Therefore, each element is removed with probability at most $(2eb)^{W-1}$ for each constraint where it participates. \square

We remark that a k -sparse packing system can be viewed as the intersection of multiple knapsack constraints on the elements where each element participates in at most k constraints. One can use the composition lemma (Lemma 1.6) and the CR-schemes for a single knapsack constraint given by Lemma 4.15 to obtain CR-schemes for k -sparse packing systems. The schemes that we described and analyzed above can be seen as direct implementations of the composition approach.

4.7 UFP in paths and trees

We consider the following routing/packing problem. Let $T = (V, E)$ be a capacitated tree with u_e denoting the capacity of edge $e \in E$. We are given k distinct node pairs $s_1 t_1, \dots, s_k t_k$ with pair i having a non-negative demand d_i . We assume that the instance satisfies the no-bottleneck condition, that is, $d_{\max} = \max_i d_i \leq u_{\min} = \min_e u_e$. We say that an instance is a unit-demand instance if $d_i = 1$ for each $i \in N$ and u_e is a non-negative integer for each $e \in E$.

Let $N = \{1, \dots, k\}$, and for $i \in N$, we denote by $Q_i \subseteq E$ the edges on the unique path between s_i and t_i in T . We say that $S \subseteq N$ is *routable* if, when routing d_i units of flow from s_i to t_i over Q_i for each $i \in S$, then the total flow on any edge e is at most u_e . More formally, S is routable if

$$\sum_{i \in S: e \in Q_i} d_i \leq u_e \quad \forall e \in E.$$

We are interested in finding a routable set $S \subseteq N$ that maximizes some weight function on N . The case of linear weights was considered in [14]. Here, a weight $w_i \geq 0$ is given for $i \in N$, and the goal is to find a routable set $S \subseteq N$ that maximizes $\sum_{i \in S} w_i$. A constant factor approximation has been presented for this problem [14], and moreover it is known that the problem is APX-hard even for unit-demands and unit-weights [27].

We are interested in more general submodular weights. Let $\mathcal{I} = \{S \subseteq N \mid S \text{ is routable}\}$. The problem we consider is $\max_{S \in \mathcal{I}} f(S)$, where f is a given non-negative submodular function. We present a CR scheme for this problem that implies a constant factor approximation through our framework. We start by presenting a CR scheme for unit demands, which we then extend to general demands.

A natural (packing) LP relaxation for $P_{\mathcal{I}}$ has a variable $x_i \in [0, 1]$ for each pair i and a constraint $\sum_{i: e \in Q_i} d_i x_i \leq u_e$ for each edge e ; recall that Q_i is the set of edges on the unique s_i - t_i path in T .

CR scheme for unit-demands:

- Root T arbitrarily. Let depth of pair $s_i t_i$ be the depth of the least common ancestor of s_i and t_i in T .
- Let $R \subseteq N$ be random set obtained by including each i independently with probability bx_i .
- Let $I = \emptyset$.
- Consider pairs in R in increasing order of depth.
 - Add i to I if $I \cup \{i\}$ is routable, otherwise reject i .
- Output I .

The techniques in [9, 14] give the following lemma.

Lemma 4.19. *For any $b \in (0, \frac{1}{3e})$ the above is a $(b, 1 - \frac{2eb}{1-eb})$ -balanced CR scheme.*

Proof. Let $\mathbf{x} \in b \cdot P_{\mathcal{T}}$. Consider a fixed pair i^* and let v be the least common ancestor of s_{i^*} and t_{i^*} in the rooted tree T ; note that v could be one of s_{i^*} or t_{i^*} . Let P be the unique path in T from v to s_{i^*} and P' be the path from v to t_{i^*} . Without loss of generality assume that $v \neq s_{i^*}$ and hence P is non-empty. We wish to upper bound $\Pr[i^* \notin I \mid i^* \in R]$, that is, the probability that i^* is rejected conditioned on it being included in the random set R . The reason that i^* gets rejected is that at least one edge $e \in P \cup P'$ is already full from the pairs that have been accepted into I prior to considering i^* . We upper bound the probability of this event happening for some edge in P and use a symmetric argument for P' .

Let e_1, e_2, \dots, e_h be the edges in P from v to s_{i^*} . Let \mathcal{E}_j be the event that i^* gets rejected at e_j , that is, the capacity of e_j is full when i^* is considered for addition to I . Note that these events are correlated. We claim the following: if $j > h$ and $u_{e_j} \geq u_{e_h}$ then \mathcal{E}_j happens only if \mathcal{E}_h happens. The reason for this is the order in which the pairs in R are considered for insertion. When i^* is considered, the only pairs inserted in I prior to it are those whose depth is no larger, and hence the total capacity used on an edge decreases as we traverse the path P from v to s_i . Thus, to analyze the probability of rejection it suffices to consider a subsequence of e_1, e_2, \dots, e_h starting with e_1 such that the capacity of the next edge in the sequence is strictly smaller than the previously added one. For notational simplicity we will therefore assume that $u_{e_1} > u_{e_2} > \dots > u_{e_h} \geq 1$.

Let $S_j = \{i \neq i^* \mid e \in Q_i\}$ be the set of pairs other than i^* that contain e in their path Q_i . Let \mathcal{E}'_j be the event that $|R \cap S_j| \geq u_{e_j}$. It is easy to see that $\Pr[\mathcal{E}_j] \leq \Pr[\mathcal{E}'_j]$. Since $\frac{1}{b}\mathbf{x}$ is a feasible solution to the LP relaxation we have $\sum_{i \in S_j} x_i < bu_{e_j}$. Letting X_i be the event that $i \in R$, and $X = \sum_{i \in S_j} X_i$, we have $\Pr[\mathcal{E}'_j] = \Pr[X \geq u_{e_j}]$. Since X is the sum of independent $[0, 1]$ random variables X_i , and has expectation bu_{e_j} , we obtain by standard Chernoff bounds:

$$\Pr[\mathcal{E}'_j] = \Pr[X \geq u_{e_j}] \leq (e^\delta / (1 + \delta)^{1+\delta})^\mu \leq (e / (1 + \delta))^{(1+\delta)\mu},$$

where $\mu = bu_{e_j}$ and $\delta = 1/b - 1$. Hence, $\Pr[\mathcal{E}'_j] \leq (eb)^{u_{e_j}}$. Taking the union bound over all edges in the path, the probability of rejection of i^* on some edge in P is at most $\sum_{j=1}^h (eb)^{u_{e_j}} \leq \sum_{\ell=1}^{\infty} (eb)^\ell = \frac{eb}{1-eb}$, where the inequality is due to the fact that the edge capacities are strictly decreasing and lower bounded by 1, and the equality is due to the fact that $eb < 1$ (recall that $b \in (0, \frac{1}{3e})$). By a union bound over P and P' we have that the probability of i^* being rejected conditioned on it being in R is at most $\frac{2eb}{1-eb}$. \square

CR scheme for general demands: A CR scheme for general demands can be obtained as follows. The linear program $P_{\mathcal{T}}$ is a packing LP of the form $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \in [0, 1]$ where A is column-restricted (all the non-zero values in a column have the same value). For such column-restricted packing integer programs (CPIPs), when demands satisfy the no-bottleneck assumption, one can use grouping and scaling techniques first suggested by Kolliopoulos and Stein [31] (see also [14]) to show that the integrality gap for a CPIP with matrix A is at most a fixed constant factor worse than that of the underlying 0-1 matrix A' (obtained from A by placing a 1 in each non-zero entry). Note that in the context of the UFP problem, the matrix A corresponds to the problem with arbitrary demands while the matrix A' corresponds to the one with unit-demands. One can use the same grouping and scaling techniques to show that a monotone $(b, 1 - b')$ -balanced CR scheme for A' can be used to obtain a monotone $(b/6, (1 - b')/2)$ -balanced CR scheme for A . We give a proof in Section 4.8, see Theorem 4.20. Using this general conversion theorem and Lemma 4.19, one can obtain a (b, b') -balanced CR scheme for UFP in trees for some sufficiently small but absolute constants b and b' . This suffices

to obtain a constant factor approximation for maximizing a non-negative submodular function of routable requests in a capacitated tree. However, the $(b/6, (1 - b')/2)$ -balanced CR scheme does not allow composition with other constraints via Lemma 1.6 since $(1 - b')/2$ does not tend to zero even if b' does. However, Theorem 4.20 gives a more refined statement that is helpful in applications in light of Remark 1.8.

Without the no-bottleneck assumption, the linear program has an $\Omega(n)$ integrality gap even for UFP on paths [9]. One can still apply the grouping and scaling techniques without the no-bottleneck assumption under a mild restriction; we refer the reader to [13].

4.8 Column-restricted packing constraints

Here we consider CR schemes for CIPs. We follow the notation from [14]. Let A be an arbitrary $m \times n$ $\{0, 1\}$ -matrix, and d be an n -element non-negative vector with d_j denoting the j th entry in d . Let $A[d]$ denote the matrix obtained by multiplying every entry of column j in A by d_j . A CIP is a problem of the form $\max w\mathbf{x}$, subject to $A[d]\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \in \{0, 1\}^n$. Note that all non-zero entries in $A[d]$ for any given column have the same value and hence the name column-restricted. Here we are interested in submodular objective functions and the goal is obtain a CR scheme for the polytope $P_{\mathcal{I}}$ induced by the relaxation $A[d]\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \in [0, 1]^n$. Instead of focusing on the polytope for a given d and b , we consider the class of polytopes induced by all d, b .

Theorem 4.20. *Suppose there is a monotone $(\beta, 1 - \beta')$ CR scheme for the polytope $A\mathbf{x} \leq \mathbf{b}$, $x \in [0, 1]^n$ for every $\mathbf{b} \in \mathbb{Z}_+$ where A is $\{0, 1\}$ -matrix. Then there is a monotone $(\beta/6, (1 - \beta')/2)$ -balanced CR scheme for the polytope $A[d]\mathbf{x} \leq \mathbf{b}$, $x \in [0, 1]^n$ for all d, b such that $d_{\max} = \max_j d_j \leq b_{\min} = \min_j b_j$. Moreover there is a monotone $(\beta/6, 1 - \beta')$ -balanced CR scheme if all $d_j \leq b_{\min}/3$ or if all $d_j \geq b_{\min}/3$.*

We sketch the proof of the above theorem which follows the grouping and scaling ideas previously used in [31, 14]. We have chosen some specific constants in the theorem for simplicity. One can obtain some generalizations and variations of the above theorem via the same ideas.

Let $N = \{1, \dots, n\}$ be a ground set corresponding to the columns. Given d , for integer $h \geq 0$ we let $N_h = \{j \in N \mid d_j \in (d_{\max}/3^{h+1}, d_{\max}/3^h]\}$. We think of the columns in N_0 as *large* and the rest as *small*. The overall idea is to focus either on the large demands or the small demands. Moreover, we will see that small demands can be treated independently within each group N_h . Let \mathbf{z} be a feasible solution to the system $A[d]\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \in [0, 1]^n$. For integer $h \geq 0$ we let \mathbf{z}^h denote the vector obtained from \mathbf{z} as follows: $\mathbf{z}_j^h = \mathbf{z}_j/6$ if $j \in N_h$ and $\mathbf{z}_j^h = 0$ otherwise. The vector \mathbf{z}^h restricts the solution \mathbf{z} to elements in N_h and scales it down by a small constant factor. We also define a corresponding vector \mathbf{b}^h where $b_i^h = \lceil A_i \mathbf{z}^h \rceil$ for each row i . We have the following lemma which is a restatement of corresponding statements from [31, 14].

Lemma 4.21. *For $h \geq 0$, let $\mathbf{y}^h \in \{0, 1\}^n$ be a feasible integral solution to $A\mathbf{x} \leq \mathbf{b}^h$, $\mathbf{x} \in [0, 1]^n$ such that $\mathbf{y}_j = 0$ if $\mathbf{z}_j^h = 0$. Then $A[d]\mathbf{y}^0 \leq \mathbf{b}$ and $\sum_{h \geq 1} A[d]\mathbf{y}^h \leq \mathbf{b}$.*

Proof. Fix some h and consider the i -th row of $A[d]\mathbf{y}^h$ which is equal to $\sum_{j \in N_h} d_j A_{ij} y_i^h$. We upper bound this quantity as follows:

$$\begin{aligned}
\sum_{j \in N_h} d_j A_{ij} y_i^h &\leq \frac{d_{\max}}{3^h} \sum_{j \in N_h} A_{ij} y_i^h && \text{(from definition of } N_h) \\
&\leq \frac{d_{\max}}{3^h} b_i^h && \text{(feasibility of } \mathbf{y}^h) \\
&\leq \frac{d_{\max}}{3^h} \left(1 + \sum_{j \in N_h} A_{ij} z_j^h \right) && \text{(definition of } \mathbf{b}^h \text{ and using } [a] \leq 1 + a) \\
&\leq \frac{d_{\max}}{3^h} \left(1 + \sum_{j \in N_h} A_{ij} z_j / 6 \right) && \text{(from definition of } \mathbf{z}^h) \\
&\leq \frac{d_{\max}}{3^h} + \frac{1}{2} \sum_{j \in N_h} A_{ij} d_j z_j && (d_j > d_{\max}/3^{h+1} \text{ for } j \in N_h).
\end{aligned}$$

For $h = 0$ we need a slight variant of the above where we replace \mathbf{b}_i^h by $\max\{1, 2 \sum_{j \in N_0} A_{ij} z_j^h\}$ since $\lceil a \rceil \leq \max\{1, 2a\}$. Then we obtain that

$$\sum_{j \in N_0} d_j A_{ij} y_i^0 \leq \max\{d_{\max}, \sum_{j \in N_0} A_{ij} d_j z_j\} \leq b_i,$$

since $d_{\max} \leq b_{\min}$ and \mathbf{z} is feasible. Thus $A[d]\mathbf{y}^0 \leq \mathbf{b}$.

For the second part of the claim, consider a row i .

$$\begin{aligned} \sum_{h \geq 1} \sum_{j \in N_h} d_j A_{ij} y_i^h &\leq \sum_{h \geq 1} \left(\frac{d_{\max}}{3^h} + \frac{1}{2} \sum_{j \in N_h} A_{ij} d_j z_j \right) \\ &\leq \sum_{h \geq 1} \frac{d_{\max}}{3^h} + \sum_{h \geq 1} \frac{1}{2} \sum_{j \in N_h} A_{ij} d_j z_j \\ &\leq \frac{d_{\max}}{2} + \frac{b_i}{2} \\ &\leq b_i. \end{aligned}$$

The penultimate inequality is from the feasibility of \mathbf{z} , and the last inequality is from the assumption that $d_{\max} \leq b_{\min}$. \square

With the above claim in place we can describe the CR scheme claimed in the theorem. Let \mathbf{z} be a feasible solution and let \mathbf{z}^h for $h \geq 0$ be constructed from \mathbf{z} as described above.

CR scheme:

- For each $h \geq 0$ *independently* run the $(\beta, 1 - \beta')$ -balanced CR scheme for the polytope $A\mathbf{x} \leq \mathbf{b}^h$, $x \in [0, 1]^n$ with fractional solution \mathbf{z}^h to obtain integral vectors \mathbf{y}^h , $h \geq 0$.
- With probability $1/2$ output \mathbf{y}^0 , otherwise output $\sum_{h \geq 1} \mathbf{y}^h$.

We claim that the above scheme is a monotone $(\beta/6, (1 - \beta')/2)$ -balanced CR scheme. Note that we use the unit-demand scheme in a black-box fashion. First, we observe via Lemma 4.21 that the output of the scheme is a feasible integral solution. An alternative description of the scheme is as follows. We are given a point $\mathbf{x} = \frac{\beta}{6} \mathbf{z}$ with $\mathbf{z} \in [0, 1]^n$, $A\mathbf{z} \leq \mathbf{b}$. Obtain a set $R \subseteq N$ by independently sampling each $j \in N$ with probability $x_j = \beta/6 \cdot z_j$. Let $R_h = R \cap N_h$. For each h obtain $I_h \subseteq R_h$ as the output of the scheme for $A\mathbf{y} \leq \mathbf{b}^h$, $\mathbf{y} \in [0, 1]^n$ given the random set R_h . With probability $1/2$ output $I = I_0$ otherwise output $I = \cup_{h \geq 1} I_h$. For $j \in N_h$ we have that $\Pr[j \in I_h \mid j \in R_h] \geq 1 - \beta'$. Further, $\Pr[j \in I \mid j \in I_h] = 1/2$ by the choice of the algorithm in the second step. Therefore $\Pr[j \in I \mid j \in R] \geq (1 - \beta')/2$. It is easy to verify the scheme is monotone.

Further, if we only have large demands or only small demands then the second step is not necessary and hence we obtain a $(\beta/6, (1 - \beta'))$ -balanced CR scheme.

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A Approximation for general polytopes

In this section, we formulate an approximation result for the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ when P is a general solvable polytope (not necessarily down-monotone). This result is included only for the sake of completeness; we do not have any concrete applications for it. Our result generalizes (while losing a factor of 4) the result for matroid base polytopes from [50], which states that a $\frac{1}{2}(1 - \frac{1}{\nu} - o(1))$ -approximation can be achieved, provided that the fractional base packing number is at least ν where $\nu \in [1, 2]$. As observed in [50], the fractional base packing number being at least ν is equivalent to the condition $P \cap [0, \frac{1}{\nu}]^N \neq \emptyset$. This is the condition we use for general polytopes. We state the algorithm only in its continuous form; we omit the discretization details.

Algorithm A.1. Let $t \in [0, 1]$ be a parameter such that $P \cap [0, t]^N \neq \emptyset$. Initialize $\mathbf{x} \in P \cap [0, t]^N$ arbitrarily. As long as there is $\mathbf{y} \in P \cap [0, \frac{1}{2}(1+t)]^N$ such that $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) > 0$ (which can be found by linear programming), move \mathbf{x} continuously in the direction $\mathbf{y} - \mathbf{x}$. If there is no such $\mathbf{y} \in P \cap [0, \frac{1}{2}(1+t)]^N$, return \mathbf{x} .

Note that even though we require $P \cap [0, t]^N \neq \emptyset$, the local search works inside a larger polytope $P \cap [0, \frac{1}{2}(1+t)]^N$. This is necessary for the analysis.

Theorem A.2. For any solvable polytope such that $P \cap [0, t]^N \neq \emptyset$, Algorithm A.1 approximates the problem $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$ within a factor of $\frac{1}{8}(1-t)$.

Proof. The algorithm maintains the invariant $\mathbf{x} \in P \cap [0, \frac{1}{2}(1+t)]^N$. Suppose that the algorithm returns a point \mathbf{x} . Then we know that for every $\mathbf{y} \in P \cap [0, \frac{1}{2}(1+t)]^N$, $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \leq 0$. We use a particular point \mathbf{y} defined as follows: Let \mathbf{x}^* be the optimum, i.e. $F(\mathbf{x}^*) = \max\{F(\mathbf{x}) : \mathbf{x} \in P\}$, and let \mathbf{x}_0 be any point in $P \cap [0, t]^N$, for example the starting point. Then we define $\mathbf{y} = \frac{1}{2}(\mathbf{x}_0 + \mathbf{x}^*)$. By convexity, we have $\mathbf{y} \in P$, and since $\mathbf{x}^* \in [0, 1]^N$, we also have $\mathbf{y} \in [0, \frac{1}{2}(1+t)]^N$. Therefore, by the local-search condition, we have $(\mathbf{y} - \mathbf{x}) \cdot \nabla F(\mathbf{x}) \leq 0$. By Lemma 3.2,

$$2F(\mathbf{x}) \geq F(\mathbf{x} \vee \mathbf{y}) + F(\mathbf{x} \wedge \mathbf{y}) \geq F(\mathbf{x} \vee \mathbf{y}).$$

Let $\mathbf{x}' = \mathbf{x} \vee \mathbf{y}$. The point \mathbf{x}' has the following properties: $\mathbf{x}' = \mathbf{x} \vee \frac{1}{2}(\mathbf{x}_0 + \mathbf{x}^*) \geq \frac{1}{2}\mathbf{x}^*$, and also $\mathbf{x}' \in [0, \frac{1}{2}(1+t)]^N$. Consider the ray $\frac{1}{2}\mathbf{x}^* + \xi(\mathbf{x}' - \frac{1}{2}\mathbf{x}^*)$ parameterized by $\xi \geq 0$. Observe that this ray has a positive direction in all coordinates, and it is possible to go beyond $\xi = 1$ and still stay inside $[0, 1]^N$: in particular, for $\xi = \frac{2}{1+t}$ we get a point $\frac{1}{2}\mathbf{x}^* + \frac{2}{1+t}(\mathbf{x}' - \frac{1}{2}\mathbf{x}^*) \leq \frac{2}{1+t}\mathbf{x}' \in [0, 1]^N$. Using this fact, we can express \mathbf{x}' as a convex combination:

$$\mathbf{x}' = \frac{1+t}{2} \cdot \left(\frac{1}{2}\mathbf{x}^* + \frac{2}{1+t}(\mathbf{x}' - \frac{1}{2}\mathbf{x}^*) \right) + \frac{1-t}{2} \cdot \frac{1}{2}\mathbf{x}^*$$

(the reader can verify that this is an identity). By the concavity of F in positive directions, we get

$$F(\mathbf{x}') \geq \frac{1+t}{2}F\left(\frac{1}{2}\mathbf{x}^* + \frac{2}{1+t}(\mathbf{x}' - \frac{1}{2}\mathbf{x}^*)\right) + \frac{1-t}{2}F\left(\frac{1}{2}\mathbf{x}^*\right).$$

As we argued, $\frac{1}{2}\mathbf{x}^* + \frac{2}{1+t}(\mathbf{x}' - \frac{1}{2}\mathbf{x}^*) \in [0, 1]^N$, so we can just lower-bound the respective value by 0, and we obtain

$$F(\mathbf{x}') \geq \frac{1-t}{2}F\left(\frac{1}{2}\mathbf{x}^*\right) \geq \frac{1-t}{4}F(\mathbf{x}^*).$$

Finally, our solution satisfies

$$F(\mathbf{x}) \geq \frac{1}{2}F(\mathbf{x} \vee \mathbf{y}) = \frac{1}{2}F(\mathbf{x}') \geq \frac{1-t}{8}F(\mathbf{x}^*) = \frac{1-t}{8}\text{OPT}.$$

□

B Missing proofs of Section 4

Proof of Theorem 4.1

Proof. As observed in the proof of Theorem (1.3), it suffices to show (10) (assuming an arbitrary ordering of the elements $N = \{1, \dots, n\}$). Let us take the expectation in two steps, first over I conditioned on R , and then over R :

$$\begin{aligned} \mathbf{E}[f(I \cap [i]) - f(I \cap [i-1])] &\geq \mathbf{E}_R[\mathbf{E}_I[\mathbf{1}_{i \in I} f_{R \cap [i-1]}(i) \mid R]] \\ &= \mathbf{E}_R[\Pr[i \in I \mid R] f_{R \cap [i-1]}(i)]. \end{aligned}$$

Note that $\Pr[i \in I \mid R]$ can be nonzero only if $i \in R$, therefore we can restrict our attention to this event:

$$\mathbf{E}[f(I \cap [i]) - f(I \cap [i-1])] \geq \Pr[i \in R] \cdot \mathbf{E}[\Pr[i \in I \mid R] f_{R \cap [i-1]}(i) \mid i \in R].$$

On the product space associated with the distribution of R conditioned on $i \in R$, both $\Pr[i \in I \mid R]$ and $f_{R \cap [i-1]}(i)$ are non-increasing functions, due to I being monotone with respect to R , and f being submodular. Therefore, the FKG inequality (see [3]) implies that

$$\begin{aligned} \mathbf{E}_R[\Pr[i \in I \mid R] f_{R \cap [i-1]}(i) \mid i \in R] &\geq \mathbf{E}_R[\Pr[i \in I \mid R] \mid i \in R] \cdot \mathbf{E}_R[f_{R \cap [i-1]}(i) \mid i \in R] \\ &= \Pr[i \in I \mid i \in R] \cdot \mathbf{E}[f_{R \cap [i-1]}(i)]. \end{aligned}$$

since the marginal value $f_{R \cap [i-1]}(i)$ does not depend on $i \in R$. By the (b, c) -balanced property, $\Pr[i \in I \mid i \in R] \geq c$; in addition, f is either monotone or we assume that $\Pr[i \in I \mid i \in R] = c$. In both cases, $\Pr[i \in I \mid i \in R] \cdot \mathbf{E}[f_{R \cap [i-1]}(i)] \geq c \cdot \mathbf{E}[f_{R \cap [i-1]}(i)]$. We summarize:

$$\begin{aligned} \mathbf{E}[f(I \cap [i]) - f(I \cap [i-1])] &\geq \Pr[i \in R] \cdot c \mathbf{E}[f_{R \cap [i-1]}(i)] \\ &= c \mathbf{E}[f(R \cap [i]) - f(R \cap [i-1])]. \end{aligned}$$

Therefore,

$$\mathbf{E}[f(I)] = f(\emptyset) + \sum_{i=1}^n \mathbf{E}[f(I \cap [i]) - f(I \cap [i-1])] \geq f(\emptyset) + c \sum_{i=1}^n \mathbf{E}[f(R \cap [i]) - f(R \cap [i-1])] \geq c \mathbf{E}[f(R)].$$

□

Proof of Lemma 4.10

To prove Lemma 4.10 we use a property of submodular functions presented in [48], which is stated below as Lemma B.1.

Lemma B.1 ([48]). *Let $f : 2^N \rightarrow \mathbb{R}_+$ be a monotone submodular function, and $A_1, \dots, A_m \subseteq N$. For each $j \in [m]$ independently, sample a random subset $A_j(q_j)$ which contains each element of A_j with probability q_j . Let J be a random subset of $[m]$ containing each element $j \in [m]$ independently with probability q_j . Then*

$$\mathbf{E}[f(A_1(q_1) \cup \dots \cup A_m(q_m))] \geq \mathbf{E} \left[f \left(\bigcup_{j \in J} A_j \right) \right].$$

Lemma B.2 below is a generalization of Lemma 4.2 in [48]. We follow the same proof technique as used in [48]. The lemma contains two statements. The first is a simpler statement that may be of independent interest. The second, which can be seen to be a slightly stronger version of the first statement, turns out to imply Lemma (4.10), as we will show in the following.

Lemma B.2. *Let $f : 2^N \rightarrow \mathbb{R}_+$ be a monotone submodular function, and $A_1, \dots, A_m \subseteq N$. For each $j \in [m]$ independently sample a random subset $A_j(q_j)$ which contains each element of A_j with probability q_j . Let $q = \sum_{j=1}^m q_j$. Then*

$$\mathbf{E}[f(A_1(q_1) \cup \dots \cup A_m(q_m))] \geq \frac{1}{q} \left(1 - \left(1 - \frac{q}{m} \right)^m \right) \sum_{j=1}^m q_j f(A_j).$$

Furthermore for $m \geq 2$ and any $s, t \in [m]$ with $s \neq t$,

$$\begin{aligned} & \mathbf{E}[f(A_1(q_1) \cup \dots \cup A_m(q_m))] \\ & \geq \frac{1}{q - q_s q_t} \left(1 - \left(1 - \frac{q - q_s q_t}{m - 1} \right)^{m-1} \right) \left(-q_s q_t \min\{f(A_s), f(A_t)\} + \sum_{j=1}^m q_j f(A_j) \right). \end{aligned}$$

Proof. Observe that the first statement is a consequence of the second one: it suffices to add an arbitrary additional set A_{m+1} with probability $q_{m+1} = 0$ to the family of sets and invoke the second part of the lemma with $s = 1$ and $t = m + 1$. Hence, we only prove the second part of the lemma.

By Lemma B.1 it suffices to estimate $\mathbf{E}[f(\cup_{j \in J} A_j)]$, where J is a random subset of $[m]$ containing element $j \in [m]$ independently of the others with probability q_j . Assume $f(A_1) \geq \dots \geq f(A_m)$ and without loss of generality we assume $t > s$. We define for $k \in [m]$,

$$\mathcal{J}_k = \{I \subseteq [m] \mid \min(I) = k\}.$$

By monotonicity of f , we have $f(\cup_{j \in J} A_j) \geq f(A_k)$ if $J \in \mathcal{J}_k$. Hence,

$$\mathbf{E}[f(\cup_{j \in J} A_j)] \leq \sum_{j=1}^m \Pr[J \in \mathcal{J}_j] f(A_j) = \sum_{j=1}^m f(A_j) q_j \prod_{\ell=1}^{j-1} (1 - q_\ell).$$

Thus, it suffices to prove

$$\sum_{j=1}^m f(A_j) q_j \prod_{\ell=1}^{j-1} (1 - q_\ell) \geq \frac{1}{q - q_s q_t} \left(1 - \left(1 - \frac{q - q_s q_t}{m - 1} \right)^{m-1} \right) \left(-q_s q_t f(A_t) + \sum_{j=1}^m q_j f(A_j) \right). \quad (16)$$

Since the above inequality is linear in the parameters $f(A_j)$, it suffices to prove it for the special case $f(A_1) = \dots = f(A_r) = 1$ and $f(A_{r+1}) = \dots = f(A_m) = 0$ (A general decreasing sequence of $f(A_j)$ can be obtained as a positive linear combination of such special cases.) Hence, it remains to prove

$$\sum_{j=1}^r q_j \prod_{\ell=1}^{j-1} (1 - q_\ell) \geq \frac{1}{q - q_s q_t} \left(1 - \left(1 - \frac{q - q_s q_t}{m - 1} \right)^{m-1} \right) \left(-\mathbf{1}_{r \geq t} \cdot p_s p_t + \sum_{j=1}^r q_j \right), \quad (17)$$

where $\mathbf{1}_{r \geq t}$ is equal to 1 if $r \geq t$ and 0 otherwise. To prove (17) we distinguish two cases depending on whether $r < t$ or $r \geq t$.

Case $r < t$: Expanding the left-hand side of (17), we obtain

$$\sum_{j=1}^r q_j \prod_{\ell=1}^{j-1} (1 - q_\ell) = 1 - \prod_{j=1}^r (1 - q_j) \geq 1 - \left(1 - \frac{1}{r} \sum_{j=1}^r q_j\right)^r,$$

using the arithmetic-geometric mean inequality. Finally, using concavity of $\phi_r(x) = 1 - (1 - \frac{x}{r})^r$ and $\phi_r(0) = 0$, we get

$$\begin{aligned} 1 - \left(1 - \frac{1}{r} \sum_{j=1}^r q_j\right)^r &= \phi_r\left(\sum_{j=1}^r q_j\right) \geq \phi_r(q - q_s q_t) \frac{\sum_{j=1}^r q_j}{q - q_s q_t} \geq \phi_{m-1}(q - q_s q_t) \frac{\sum_{j=1}^r q_j}{q - q_s q_t} \\ &= \frac{1}{q - q_s q_t} \left(1 - \left(1 - \frac{q - q_s q_t}{m-1}\right)^{m-1}\right) \sum_{j=1}^r q_j, \end{aligned}$$

where the last inequality follows from the fact $\phi_r(x)$ is decreasing in r and by using $r \leq m-1$. Notice that we used the fact $\sum_{j=1}^r q_j \leq q - q_s q_t$ for the first inequality in the reasoning above, which holds since $r < t$ and therefore $\sum_{j=1}^r q_j \leq q - q_t \leq q - q_s q_t$.

Case $r \geq t$: As in the previous case we start by expanding the left-hand side of 17. This time we bundle the two terms $(1 - q_s)$ and $(1 - q_t)$ when applying the arithmetic-geometric mean inequality.

$$\begin{aligned} \sum_{j=1}^r q_j \prod_{\ell=1}^{j-1} (1 - q_\ell) &= 1 - \prod_{j=1}^r (1 - q_j) = 1 - (1 - q_s)(1 - q_t) \prod_{\substack{j \in [r] \\ j \notin \{s, t\}}} (1 - q_j) \\ &\geq 1 - \left(1 - \frac{1}{r-1} \left(-q_s q_t + \sum_{j=1}^r q_j\right)\right)^{r-1}. \end{aligned}$$

The remaining part of the proof is similar to the previous case.

$$\begin{aligned} 1 - \left(1 - \frac{1}{r-1} \left(-q_s q_t + \sum_{j=1}^r q_j\right)\right)^{r-1} &= \phi_{r-1}\left(-q_s q_t + \sum_{j=1}^r q_j\right) \geq \phi_{r-1}(q - q_s q_t) \frac{-q_s q_t + \sum_{j=1}^r q_j}{q - q_s q_t} \\ &\geq \phi_{m-1}(q - q_s q_t) \frac{-q_s q_t + \sum_{j=1}^r q_j}{q - q_s q_t} \\ &= \frac{1}{q - q_s q_t} \left(1 - \left(1 - \frac{q - q_s q_t}{m-1}\right)^{m-1}\right) \left(-p_s p_t + \sum_{j=1}^r q_j\right), \end{aligned}$$

again using concavity of ϕ_{r-1} and the fact that $\phi_{r-1}(q - q_s q_t)$ is decreasing in r . This finishes the proof of (17) and thus completes the proof of the lemma. \square

Leveraging Lemma B.2 we are now ready to prove Lemma 4.10. We recall that for a nonnegative submodular function $f : 2^N \rightarrow \mathbb{R}_+$ and $\mathbf{p} \in [0, 1]^N$, its concave closure f^+ is defined by

$$f^+(\mathbf{p}) = \max \left\{ \sum_{S \subseteq N} \alpha_S f(S) \mid \alpha_S \geq 0 \forall S \subseteq N, \sum_{S \subseteq N} \alpha_S = 1, \sum_{S \subseteq N, i \in S} \alpha_S = p_i \forall i \in N \right\}.$$

Proof of Lemma 4.10. Consider a basic solution $(\alpha_j, A_j)_{j \in [m]}$ to the linear program that defines $f^+(\mathbf{p})$, i.e., $f^+(\mathbf{p}) = \sum_{j=1}^m \alpha_j f(A_j)$, with $A_j \subseteq N$, $\alpha_j \geq 0$ for $j \in [m]$, $\sum_{j=1}^m \alpha_j = 1$ and $\sum_{j \in [m], i \in A_j} \alpha_j = p_i$ for $i \in N$. Notice that since we chose a basic solution and the LP defining $f^+(\mathbf{p})$ only has $n + 1$ constraints apart from the nonnegativity constraints, we have $m \leq n + 1$. Let $R(b\mathbf{p})$ be a random subset of N containing each element $i \in N$ independently with probability bp_i . We distinguish two cases depending on whether $m \leq n$ or $m = n + 1$.

Case $m \leq n$: Consider the random set

$$A = \bigcup_{j \in [m]} A_j(b \cdot \alpha_j),$$

where $A_j(b\alpha_j)$ is a random subset of N containing each element $i \in N$ with probability $b\alpha_j$, independently of the others. Notice that the distribution of A is dominated by the distribution of $R(b\mathbf{p})$ since A contains each element $i \in N$ independently with probability

$$\Pr[i \in A] = 1 - \prod_{\substack{j \in [m] \\ i \in A_j}} (1 - b\alpha_j) \leq 1 - \left(1 - \sum_{\substack{j \in [m] \\ i \in A_j}} b\alpha_j\right) = bp_i = \Pr[i \in R(b\mathbf{p})].$$

Hence $F(b\mathbf{p}) \geq \mathbf{E}[f(A)]$, and we can use the first statement of Lemma B.2 to obtain

$$F(b\mathbf{p}) \geq \mathbf{E}[f(A)] \geq \frac{1}{\sum_{j=1}^m b\alpha_j} \left(1 - \left(1 - \frac{\sum_{j=1}^m b\alpha_j}{m}\right)^m\right) \sum_{j=1}^m b\alpha_j f(A_j) \geq \left(1 - \left(1 - \frac{b}{m}\right)^m\right) f^+(\mathbf{p}),$$

using $\sum_{j=1}^m \alpha_j = 1$ and the fact that $\frac{1 - (1 - \frac{x}{m})^m}{x}$ is decreasing in x . The proof of this case is completed by observing that $(1 - (1 - \frac{b}{m})^m)$ is decreasing in m and $m \leq n$.

Case $m = n + 1$: Since $A_j \subseteq N$ for $j \in [n + 1]$ and $|N| = n$, there must be at least one set A_t that is covered by the remaining sets, i.e., $A_t \subseteq \cup_{j \in [n+1], j \neq t} A_j$. Furthermore let $s \in [n + 1] \setminus \{t\}$ be the index minimizing $b\alpha_s$. We define probabilities q_j for $j \in [n + 1]$ as follows

$$q_j = \begin{cases} b\alpha_j & \text{if } j \neq t, \\ \frac{b\alpha_t}{1 - b\alpha_s} & \text{if } j = t. \end{cases}$$

We follow a similar approach as for the previous case by considering the random set

$$A = \bigcup_{j \in [m]} A_j(q_j).$$

Again, we first observe that A is dominated by the distribution of $R(b\mathbf{p})$. For any $i \in N \setminus A_t$ the analysis of the previous case still holds and shows $\Pr[i \in A] \leq \Pr[i \in R(b\mathbf{p})]$. Consider now an element $i \in A_t$. Let $A_t, A_{j_1}, \dots, A_{j_r}$ be all sets in the family $(A_j)_{j \in [n+1]}$ that contain i . By our choice of A_t , there is at least one other set containing i , i.e., $r \geq 1$. Using

$$(1 - q_t)(1 - q_{j_1}) = \left(1 - \frac{b\alpha_t}{1 - b\alpha_s}\right) (1 - b\alpha_{j_1}) \stackrel{\alpha_s \leq \alpha_{j_1}}{\leq} 1 - b\alpha_t - b\alpha_{j_1},$$

we obtain

$$\begin{aligned} \Pr[i \in A] &= 1 - (1 - q_t)(1 - q_{j_1}) \prod_{\ell=2}^r (1 - q_{j_\ell}) = 1 - (1 - b\alpha_t - b\alpha_{j_1}) \prod_{\ell=2}^r (1 - b\alpha_{j_\ell}) \\ &\leq 1 - \left(1 - \sum_{\substack{j \in [n+1] \\ i \in A_j}} b\alpha_j\right) = bp_i = \Pr[i \in R(b\mathbf{p})]. \end{aligned}$$

Therefore, we again have $F(b\mathbf{p}) \geq \mathbf{E}[f(A)]$. Notice that $q = \sum_{j=1}^{n+1} q_j$ satisfies

$$q = b\alpha_t - \frac{b\alpha_t}{1 - b\alpha_s} + \sum_{j=1}^{n+1} b\alpha_j = q_s q_t + \sum_{j=1}^{n+1} b\alpha_j = q_s q_t + b.$$

We apply the second statement of Lemma B.2 to the family $(A_j(q_j))_{j \in [n+1]}$ and use the above fact to obtain

$$\begin{aligned} F(b\mathbf{p}) \geq \mathbf{E}[f(A)] &\geq \frac{1}{q - q_s q_t} \left(1 - \left(1 - \frac{q - q_s q_t}{n} \right)^n \right) \left(-q_s q_t \min\{f(A_s), f(A_t)\} + \sum_{j=1}^{n+1} q_j f(A_j) \right) \\ &= \frac{1}{b} \left(1 - \left(1 - \frac{b}{n} \right)^n \right) \left(-q_s q_t \min\{f(A_s), f(A_t)\} + (q_t - b\alpha_t) f(A_t) + b f^+(\mathbf{p}) \right) \\ &\geq \frac{1}{b} \left(1 - \left(1 - \frac{b}{n} \right)^n \right) \left(-q_s q_t f(A_t) + (q_t - b\alpha_t) f(A_t) + b f^+(\mathbf{p}) \right). \end{aligned}$$

The claim follows by observing that $q_s q_t = q_t - b\alpha_t$. □

C Details in constructing CR schemes via the ellipsoid algorithm

Here we give the technical details that are involved in sampling and approximately solving (DP1) and (LP1) from Section 4.2. First a short reminder of the primal and dual problem.

$$\begin{aligned} \text{(LP1)} \quad & \max \quad c \\ & \text{s.t.} \quad \sum_{\phi \in \Phi^*} q_{i,\phi} \lambda_\phi \geq x_i c \quad \forall i \in N \\ & \quad \quad \sum_{\phi \in \Phi^*} \lambda_\phi = 1 \\ & \quad \quad \lambda_\phi \geq 0 \quad \forall \phi \in \Phi^* \\ \text{(DP1)} \quad & \min \quad \mu \\ & \text{s.t.} \quad \sum_{i \in N} q_{i,\phi} y_i \leq \mu \quad \forall \phi \in \Phi^* \\ & \quad \quad \sum_{i \in N} x_i y_i = 1 \\ & \quad \quad y_i \geq 0 \quad \forall i \in N \end{aligned}$$

We start by observing that we can obtain strong estimates of $q_{i,\phi}$ for any $\phi \in \Phi^*$. We assume that ϕ is given as an oracle and can therefore be evaluated in constant time.

Proposition C.1. *Let $\phi \in \Phi^*$ and $i \in N$. An estimate $\hat{q}_{i,\phi}$ of $q_{i,\phi}$ whose error is bounded by $\pm \epsilon x_i$ with high probability can be obtained in time polynomial in n and $\frac{1}{\epsilon}$.*

Proof. We call a set $S \subseteq N \setminus \{i\}$ *good*, if $i \in \phi(S \cup \{i\})$. We have

$$\begin{aligned} q_{i,\phi} &= \Pr[i \in \phi(R)] = \Pr[i \in R \text{ and } R \setminus \{i\} \text{ is good}] \\ &= \Pr[i \in R] \cdot \Pr[R \setminus \{i\} \text{ is good}] = x_i \cdot \Pr[R \setminus \{i\} \text{ is good}]. \end{aligned}$$

Notice that we can estimate $\Pr[R \setminus \{i\} \text{ is good}]$ up to an error of $\pm \epsilon$ with high probability by a standard Monte Carlo approach, where we draw samples of $R \setminus \{i\}$. This can be done in time polynomial in n and $\frac{1}{\epsilon}$ and leads to the claimed estimate $\hat{q}_{i,\phi}$ by the above formula. □

We now discuss how these estimates can be used to obtain a near-optimal solution to (DP1) by employing the ellipsoid method with a weak separation oracle. After that we show how a near-optimal solution to (LP1) can be obtained. Notice that Proposition C.1 can easily be used to obtain estimates $\tilde{q}_{i,\phi}$ of $q_{i,\phi}$ that satisfy with high probability $\tilde{q}_{i,\phi} \in [q_{i,\phi} - \epsilon x_i, q_{i,\phi}]$: it suffices to consider an estimate $\hat{q}_{i,\phi}$ of $q_{i,\phi}$ that satisfies with high probability $\hat{q}_{i,\phi} \in [q_{i,\phi} - \frac{\epsilon}{2} x_i, q_{i,\phi} + \frac{\epsilon}{2} x_i]$ and to define $\tilde{q}_{i,\phi} = \hat{q}_{i,\phi} - \frac{\epsilon}{2}$. In the following, we assume that all used estimates $\tilde{q}_{i,\phi}$ satisfy $\tilde{q}_{i,\phi} \in [q_{i,\phi} - \epsilon x_i, q_{i,\phi}]$. We can obtain this with high probability through Proposition C.1 since the ellipsoid

that we will apply in the following only uses a polynomial number of such estimates. Notice that these estimates are “pessimistic” estimates for $q_{i,\phi}$, i.e., replacing $q_{i,\phi}$ in (LP1) by these estimates leads to a lower optimal value of the LP.

Furthermore, to simplify the exposition, we will assume that for any given weight vector $\mathbf{y} \in \mathbb{R}_+^N$ we can find a CR scheme $\phi \in \Phi^*$ maximizing $\sum_{i \in N} q_{i,\phi} y_i$. The following discussion works also if we can only find a CR scheme ϕ that approximately maximizes this expression.

To apply the ellipsoid algorithm to (DP1) we design a weak separation oracle (see Chapter 4 in [32]). As a reminder, the weak separation oracle has to provide the following guarantees. Given is a nonnegative vector $\mathbf{y} = (y_i)_{i \in N}$ satisfying $\sum_{i \in N} x_i y_i = 1$, and a value μ . The weak separation oracle has to provide either a feasible dual solution (\mathbf{y}', μ') with $\mu' \leq \mu + \epsilon$, or a hyperplane separating (\mathbf{y}, μ) from all feasible dual solutions. Given \mathbf{y} and μ , let $\phi \in \Phi^*$ be the CR scheme that maximizes $\sum_{i \in N} q_{i,\phi} y_i$. If $\sum_{i \in N} \tilde{q}_{i,\phi} y_i \leq \mu$, our weak separation oracle returns the dual solution $(\mathbf{y}, \mu + \epsilon)$. This solution has objective value $\mu + \epsilon$ as desired and is indeed feasible since for any $\phi' \in \Phi^*$,

$$\sum_{i \in N} q_{i,\phi'} y_i \leq \sum_{i \in N} q_{i,\phi} y_i \leq \sum_{i \in N} (\tilde{q}_{i,\phi} + \epsilon x_i) y_i = \epsilon + \sum_{i \in N} \tilde{q}_{i,\phi} y_i \leq \mu + \epsilon.$$

If $\sum_{i \in N} \tilde{q}_{i,\phi} y_i > \mu$, our weak separation oracle returns the separating hyperplane given by the constraint $(\mathbf{a}, -1) \cdot (\mathbf{z}, \nu) \leq 0$ where \mathbf{a} is a n -dimensional vector with coefficients $a_i = \tilde{q}_{i,\phi}$ for $1 \leq i \leq n$ (note that that (DP1) has $n + 1$ variables corresponding to y_1, \dots, y_n and μ). First, this hyperplane indeed cuts off the solution (\mathbf{y}, μ) . Furthermore, if (\mathbf{y}', μ') is a feasible dual solution then it satisfies the constraint since $\tilde{q}_{i,\phi} \leq q_{i,\phi}$:

$$\sum_{i \in N} \tilde{q}_{i,\phi} y'_i \leq \sum_{i \in N} q_{i,\phi} y'_i \leq \mu',$$

where the second inequality in the above follows from feasibility of (\mathbf{y}', μ') . Hence, we obtained a weak separation oracle for (DP1), and the ellipsoid method can therefore determine a feasible solution (\mathbf{y}, μ) to (DP1) of value $\leq \mu^* + 2\epsilon$, where μ^* is the value of an optimal dual solution. Note that since (\mathbf{y}, μ) is feasible, we have $\mu^* \leq \mu \leq \mu^* + 2\epsilon$ (see [32]).

Let (DP1') be the linear program obtained from (DP1) by only considering constraints corresponding to CR schemes ϕ that were used in the ellipsoid algorithm while constructing the nearly optimal solution (\mathbf{y}, μ) of (DP1), which satisfies $\mu \leq \mu^* + 2\epsilon$. Furthermore, we replace all terms $q_{i,\phi}$ by their estimates $\tilde{q}_{i,\phi}$ in (DP1'). Hence, the feasible region of (DP1') consists of all separating hyperplanes that were generated during the ellipsoid algorithm. Notice that (DP1') is a relaxation of (DP1) since our estimates satisfy $\tilde{q}_{i,\phi} \leq q_{i,\phi}$. Hence, the optimal value μ' of (DP1') satisfies $\mu' \leq \mu^*$. The ellipsoid algorithm actually certifies the approximation quality of the generated solution (μ, \mathbf{y}) by comparing against the best solution satisfying the generated constraints, i.e.,

$$\mu \leq \mu' + 2\epsilon.$$

Let (LP1') be the dual of (DP1'), and let $(\boldsymbol{\lambda}', c')$ be an optimal solution to (LP1'), which can be efficiently determined since (LP1') has polynomial size. We return $(\boldsymbol{\lambda}', c')$ as the solution to (LP1). First, notice that $(\boldsymbol{\lambda}', c')$ is feasible for (LP1) since $\tilde{q}_{i,\phi} \leq q_{i,\phi}$. Furthermore,

$$c' = \mu' \geq \mu - 2\epsilon \geq \mu^* - 2\epsilon = c^* - 2\epsilon,$$

where the two equalities follow by strong duality.